TRANSMATHEMATICS

how to divide by zero using only the operations of ordinary arithmetic, but ignoring the prescription not to divide by zero, in such a way as to preserve the maximum information about the magnitude and sign of numbers

$$\Phi = \frac{0}{0}$$

James Anderson

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Contributors

The author of this book is James A.D.W. Anderson. The following people have contributed text to the book. Their contributions can be found by using the index *Authors and Historical Figures*.

Steve Leach, implementor of the Perspex compiler.

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Respect

If a man does not keep pace with his companions,
Perhaps it is because he hears a different drummer.
Let him step to the music which he hears,
However measured or far away.

From *Walden, or Life in the Woods* by
Henry David Thoreau

Preface

This book is intended to calm a storm and lead everyone to a better understanding of mathematics. Little did I know what reaction there would be when I went to an open day, at Highdown school in Reading, and showed the pupils there how to divide by zero. The presentation was televised and, within a few hours, a storm of protest blew up across the world. In the succeeding weeks, over one hundred thousand people commented on my work. This convinced me that there is a great public interest in division by zero, and a very great need to get the facts straight. There is a need, too, to make crystal clear what my views on division by zero are, and to show how division by zero can improve all our lives.

Transmathematics is just ordinary mathematics, except that it ignores the prescription not to divide by zero. It comes as a great shock to many people that this can be done without introducing contradictions into mathematics, but this is so, and was proved even before the infamous open day. If you doubt it, read the chapter on the axioms of transreal arithmetic. But it is easier to start with the chapter that explains how to do arithmetic with a pencil and paper when division by zero is allowed. Subsequent chapters on elementary algebra, limits, power series, differential calculus, integral calculus, metric spaces, and topological spaces form the main spine of the book. These chapters introduce many of the topics that are used in everyday mathematics. I have filled these out with chapters on topics that I hope will be helpful. Other chapters, including those on the nature of numbers, poles, logarithms, and the Riemann Sphere are included because I was asked for them by courteous correspondents. Finally, some chapters are included just for fun. The result is more like a cook book than a mathematical treatise – here are a thousand recipes for dividing by zero, without getting your fingers burnt!

The first part of the trick to dividing by zero has been known for a long time. I adopt the axiom that any positive number divided by zero is *positive infinity* and, correspondingly, any negative number divided by zero is *negative infinity.* But I hold that positive infinity and negative infinity are separate from each other and are not joined as an *unsigned infinity.* I have a very practical reason for assuming this. If I want a single, unsigned infinity, as is used in projective geometry, then I just glue my two signed infinities together by operating on their modulus, or absolute value. This is easy to do. But if I had adopted a single infinity, as an axiom, it would be very difficult to split it apart into positive and negative parts. Thus, I have adopted one model of infinity that mathematicians recognise as the extended, real-number line with affine (not projective) infinities.

The second part of the trick to dividing by zero is so subtle that no one seems to appreciate how it works when they first have it explained to them. Remember that any positive number divided by zero is the number positive infinity and any negative number divided by zero is the number negative infinity, but what, then, is zero divided by zero? It is the number *nullity.*

That's it. Zero divided by zero is a number. This does two amazing things. Firstly, ordinary arithmetic is nearly *total.* The sum (addition) of any two numbers is a number, the difference (subtraction) of any two numbers is a number, the product (multiplication) of any two numbers is a number, but no number can be divided by zero. So division applies only to part of the set of numbers. However, by introducing positive infinity, negative infinity and, crucially, nullity, the quotient (division) of any two numbers is a number. So the new form of division applies to the totality of the set of numbers, just like the other operations of ordinary arithmetic, and this makes *transarithmetic* total. A total arithmetic always works, which is just to say that it never fails, and this is very useful. Having computers that cannot fail when doing arithmetic is a very good thing indeed. This is the property that will make our lives better and safer. But the second property of nullity is even more amazing.

The numbers in transarithmetic are absolutely fixed and immutable, but we can use them to model situations in which we have less than perfect knowledge. We can use transarithmetic to calculate with ordinary numbers so that we know their magnitude and sign. But there are practical and mathematical occasions when numbers become so big that we cannot tell exactly how big they are – we know only that they have a very large magnitude and are positive, or that they have a very large magnitude and are negative. Transarithmetic's positive and negative infinities can stand in for these numbers when we use them as limits in calculus. Sometimes we know only that a number has a very large magnitude, but we do not know what sign it has. In these cases the modulus of transarithmetic's positive and negative infinities stands in for the unsigned, but very large, magnitude. And there is one more case, where we know nothing of the magnitude and sign of a number. Ordinary arithmetic cannot describe this situation, but transarithmetic can. Nullity stands in for a number with an unknown magnitude and sign. Thus, we may interpret *transnumbers* as modelling a hierarchy, running from complete knowledge of ordinary numbers, partial knowledge of the magnitude and, possibly, the sign of infinities, and total lack of knowledge of the magnitude and sign of zero divided by zero. In order to describe this state of affairs we have to annotate ordinary mathematics with explanations of where it does and does not work, and we have to take abortive action in programs where the underlying mathematics fails; but we do not have to do that with our new mathematics. *Transmathematics* works everywhere and each number tells us what is known about its sign and magnitude. Thus, the operations of transarithmetic provide a calculus of knowledge and ignorance. That really is amazing! Henceforth, computers can calculate what they do and do not know, just by performing arithmetic. The details of how to do this are rather more intricate than I have described here, but I give examples of how to do it throughout the book.

Some mathematicians say that nullity is no different from the bottom element, of mathematical logics, that represents a state of total ignorance, but this misses a very important point. Transarithmetic was not developed as an axiomatic system, it was developed by using the existing methods for performing ordinary arithmetic, but ignoring the prescription not to divide by zero, while preserving the maximum possible amount of information about the sign and magnitude of numbers. This leaves no room for manoeuvre in assigning properties to nullity, and therein lies the difference. We have the freedom to change the properties of *bottom,* we have no freedom to change the properties of nullity. Nullity is a fixed number.

If I wanted to be cute with the cognoscenti of mathematical logic, I would say that bottom has a successor, but nullity does not. They would smile, wryly, and admit that bottom and nullity are different things.

Now we come to the nub of the issue. Mathematicians have been able to divide by zero in arithmetic for fifty years, or more, but they do it, in various ways, by defining the infinities and a bottom element axiomatically. By contrast, I just follow ordinary arithmetic, but allow division by zero in a way which preserves the maximum information about the sign and magnitude of numbers. Consequently, everything in transmathematics is consistent with ordinary mathematics and, as a bonus, transmathematics preserves the maximum amount of information about sign and magnitude. It is the preservation of the maximum possible information that makes transarithmetic uniquely well suited to being the principal extension of ordinary arithmetic. All other extensions of ordinary arithmetic, using the extended real-numbers with a bottom element, can be obtained by accepting transarithmetic as is, or by throwing away some of the information in transarithmetic so that it matches a more limited conception of the infinities and a bottom element. Of course, some extensions to ordinary arithmetic introduce more numbers than two signed infinities and a bottom element, but it seems that transarithmetic can be extended in the same ways, while maintaining the property of preserving the maximum possible information about the sign and magnitude of numbers. If so, transarithmetic generalises any other generalisation of ordinary arithmetic. The generalisation of ordinary, real arithmetic to ordinary, complex arithmetic and its subsequent generalisation to *transcomplex* arithmetic is particularly interesting.

Transmathematics can do everything that ordinary mathematics does, and more. The critical issue is whether or not these new results are useful. I claim that they are useful in computing because they remove all arithmetical exceptions and make it easier to design processors and write programs. Both of these simplifications contribute to making computerised systems safer. I claim that transmathematics is logically consistent so it is a valid kind of pure mathematics. I may well make a mistake in developing transmathematics and introduce some inconsistency, but I expect that I will be able to correct my mistakes so that transmathematics remains consistent. In this effort, machine proof is extremely valuable. It provides exceptionally detailed checking which is easy to re-do when corrections are made. Finally, I am on the look out for physical systems that can be described more easily using transmathematics than with ordinary mathematics. If I find such systems, I will have demonstrated that transmathematics is useful in developing our scientific understanding of the universe, and this might translate into engineering applications of that knowledge, making our lives better. In conclusion, my view is that division by zero provides three things: easier and safer computation; a consistent extension of ordinary mathematics that preserves the maximum possible information about the sign and magnitude of numbers; and it creates the possibility of describing the physical universe in a way that is better than using ordinary mathematics. I hope my views are now crystal clear. If not, then ask me for a clarification.

In this preface, I have tried to give you something of the flavour of transmathematics. I hope you enjoy this book and will go on to use transmathematics in your daily life. You can use it everywhere that you use ordinary mathematics, and it will reward you by making your life safer and better.

If you have any constructive suggestions to make about this book then do contact me, or drop me a note to say what successes or difficulties you encounter in using transmathematics.

Dr James Anderson, BSc, PhD, MBCS, CITP, CSci.
Lecturer in Computer Science
Reading University
England

email: author@bookofparagon.com
web: www.bookofparagon.com

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CHAPTER 1

Introduction

Division by zero

Since the invention of zero, more than a thousand years ago, people have wanted to know how to divide by zero. For many centuries, every attempt ended in failure so that it became an assumption, ingrained by experience, that division by zero is impossible. The reasons for this failure were eventually explained by giving proofs of the impossibility of dividing by zero, but, just like the proof that it is impossible for a rocket to fly to the moon, because no chemical substance on Earth has enough energy to lift its own weight into orbit, the solution was easy enough when it was recognised in the twentieth century. In the case of rockets, burn up a lot of fuel to deliver a small payload onto the moon, in the case of division by zero, define it by axiom. But there is something unsatisfactory about the axiomatic methods for dividing by zero. Some work on abstract sets of things, using mathematical techniques far removed from the arithmetic of ordinary life. Some work on various algebraic objects, but do not preserve the properties of ordinary numbers. Each of these techniques is useful in its own sphere, but what is wanted is a method for dividing by zero using the ordinary methods of arithmetic. It will come as shock to many people that such a method exists.

I surveyed a number of the pencil and paper methods for performing arithmetic and discovered that some of these algorithms continue to give answers even when division by zero is involved. It took me a while to collect enough of these methods together to support the whole of arithmetic, but, when the job was done, I was invited to give a seminar on *transarithmetic* at Essex university, in England.

The audience knew that there was a time when people could count using only whole numbers, $1, 2, 3, \ldots$. Such ancient peoples could add and multiply any two numbers together to produce a number as a result. But they could not do the same with subtraction. It was not until zero was invented that they could subtract a number from itself and produce the answer zero. Then they could do things like $1-1 = 0$ and $2-2 = 0$. They could even do, $0-0 = 0$, but they could not subtract just any number from any other. It was not until negative numbers were invented that they could do, $1-2 = -1$. At this point, any two numbers could be added, subtracted, or multiplied to produce a number as a result, but the same could not be done with division. It took a few more centuries to make multiplication work with irrational numbers, but, throughout, division by zero seemed impossible.

And then there I was, showing a room full of scholars how to divide by zero using only the arithmetical methods we have all known for a long time, without drawing on any of the axiomatic methods for dividing by zero. During the usual to and fro of questions, one of them asked if I had a formal proof that these methods are consistent. I said that I had published a hand proof that the methods are consistent with ordinary arithmetic, but that I had not used a computer to check that the methods are internally consistent, because that would be a very great deal of work. They were a little disappointed, but one of their number volunteered to produce the proof, if I would write the methods down as axioms. I went back to my own university and wrote down the axioms. My colleague from Essex was true to his word and, in about three weeks work, spread over a couple of months, he translated the axioms into higher order logic, extended a standard model of the real numbers so that it could model the transreal numbers, proved a number of theorems that follow from the axioms, and set a computer the task of testing the axioms against this model. The computer tested every possible combination of the axioms and concluded that transarithmetic is consistent.[6]

Since then, I have continued to develop transmathematics. I have set myself the task of developing a thread of mathematics from arithmetic to topological spaces. When this is in place, I will be able to fill in transmathematics so that it deals with all of the mathematics that people commonly use. And this will be a benefit to us all. Transarithmetic, which is the basis of transmathematics, does everything that ordinary arithmetic does, and more. It is a *total* arithmetic, which is to say that every operation of arithmetic – addition, subtraction, multiplication and division – can all be applied to any numbers and the result is a number. This, as my colleagues at Essex knew, is interesting from an academic point of view, because it supplies a piece of the jigsaw that has been sought for a thousand years, and more. But, as they also knew, it would make computers quicker and safer, because there would be no possibility of arithmetical failure. The application of transarithmetic to computers will make all of our lives safer and better.

But I remain on the lookout for a bigger prize. It is conceivable that the physical world operates according to the rules of transarithmetic, not just according to the rules of ordinary arithmetic. If I can find examples of this, then transarithmetic will illuminate our understanding of the world in a way which cannot be done using ordinary arithmetic. It is a lofty goal, but a worthy one.

Sadly, some people are not prepared to consider the possibility that division by zero is possible. They read tittle-tattle about my work, and do not read the primary sources. Or they read of the machine proof of consistency and ignore it. They read the axioms and dismiss them as being inconsistent, without even trying to find an inconsistency. They say that dividing by zero must be an error, just like their forebears who might have complained that subtracting a number from itself must be an error. And they give no reason to support their beliefs. They have not learned from history.

The history of science tells us that paradigm shifts occur, in which the accepted theories, or paradigms, of the day are overturned by a new insight. During a paradigm shift it is quite normal for scientists to behave badly: to be rude, to ignore evidence, and to refuse to think about the new way of doing things. Regrettably, that is the stage we are now at. This book is not for those people. It is for the reader who is prepared to consider the possibility that numbers can be divided by zero. A possibility that lies on very firm foundations, since publication of the computer proof.[6]

In order to follow the story of division by zero from its historical roots we need little more than a grasp of the arithmetic of fractions, f, of a numerator, n, and a denominator, d, where $f = n/d$. But we shall draw on rather more mathematics as we progress to the development of transmathematics in the rest of the book.

Indian mathematics

The number zero was known in 7th century India. Brahmagupta[1] (598 - 670) correctly describes various properties of the addition, subtraction, and multiplication of zero, but says that “zero divided by zero is zero.” This is not a solution that is recognised, today, as being generally useful. In 9th century India, Brahmagupta’s ideas were updated by Mahavira[1] (circa 800 - 870) who says that “a number remains unchanged when divided by zero.” This is a very odd thing to say if it means $n/0 = n$, because division by zero and division by one then have the same effect as each other. But if the text means “a number remains unchange*able* when divided by zero," it brings us closer to a 12th Century, Indian interpretation of division by zero. Bhaskara[1] (1114 - 1185) writes:

> *A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity. In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God when worlds are created or destroyed, though numerous orders of beings are absorbed or put forth.*

My contemporaries are inclined to interpret this text as meaning $n/0 = \infty$, but dismiss this because they suppose they can multiply out the denominator, zero, to give $0 \times \infty = 0 \times \frac{n}{0} = \frac{0 \times n}{0} = n$ so that zero times infinity is equal to every number, n. I take a different line that, in hind sight, Bhaskara is very nearly correct. I say that any positive number, n, divided by zero is *positive infinity,* $n/0 = \infty$; any negative number divided by zero is *negative infinity,* $-n/0 = -\infty$; and zero divided by zero is *nullity,* $0/0 = \Phi$. I call the infinities, $\pm\infty$, *infinite numbers,* and I call nullity and the infinities *non-finite numbers.* Now I read Bhaskara as being almost correct.

> *A quantity divided by zero becomes a fraction the denominator of which is zero.*

This is true: $n \div 0 = \frac{n}{0}$

> *This fraction is termed an infinite quantity.*

No. It is termed a *non-finite* quantity.

> *In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted;*

This is nearly right. If a finite number is added to (inserted into), or subtracted from (extracted from), a non-finite number then the non-finite number is unchanged. But this leaves aside the question of what happens when non-finite numbers are added to, or subtracted from, each other. I settle this question by following the Indian mathematicians by extending the arithmetic of fractions so that it can handle a denominator of zero. In doing this I maintain all of the positive results of mathematics, but overturn some of the negative results. For example, for all finite, n , I write:

$$\frac{0 \times n}{0} = \frac{0 \times n}{0 \times n} = \frac{0}{0} = \Phi = \frac{0}{0} = \frac{0}{1} \times \frac{1}{0} = 0 \times \infty \qquad \text{[E 1.1]}$$

Thus, I overturn the view, used as a counter example, that zero times infinity is any number, $0 \times \infty = n$. Instead, I maintain that zero times infinity is the single number nullity, $0 \times \infty = \Phi$, and, though it is not shown above, that zero times negative infinity is also nullity, $0 \times (-\infty) = \Phi$. The details of how I carry this out consistently are explained in the next few chapters. But the broad outline is clear. I am following in the footsteps of the Indian mathematicians who sought to extend the arithmetic of fractions.

Overturning accepted results

The general reader may like to know that when I overturn an accepted view, this is simply a statement of mathematics. I do not imply any criticism of my colleagues. Indeed, I am very grateful to them for laying out their reasons so clearly, because it helps me to see how my work differs from theirs, and shows me where the gap in understanding arises between us. I have no doubt that, for my part, I am wrong on many matters, and that by criticising each other's work we shall close the gap between us, thereby coming to a better understanding of mathematics.

Differential calculus

There is another, broad, approach to division by zero that was pioneered in the 17th Century by Newton (1643–1727) and Leibniz (1646–1716). Differential calculus deals with the changes to a quantity that occur in the limit as a denominator is taken close to, but not equal to, zero. Calculus is astonishingly useful and is highly regarded by all, but it specifically excludes the case of division by a number that is exactly zero. Consequently, no result in calculus can contradict anything in transarithmetic, because the two areas of mathematics deal with different things. Calculus deals with division by infinitessimal numbers, excluding zero, and transarithmetic deals with division by zero. There is no overlap, and no contest, between the two approaches. I do, however, spend a great deal of time discussing limits and calculus. As always, I do not disturb any of the positive results of mathematics, but I do overturn some of the negative results. For example, I evaluate power series at essential singularities, despite the fact that calculus holds this to be impossible. I am also able to distinguish between more kinds of limits than Newton and Leibniz knew about, and I can calculate results where they could not. Thus, transarithmetic enriches our understanding of mathematics.

Denotational Semantics

Another, broad, approach to division by zero was pioneered in the 20th Century by Strachey (1916–1975) and Scott (1932–). Denotational semantics[14] explains the meaning of computer programs and uses a logical element *bottom* which means that nothing is known. Bottom can be used in many ways, and when it is added to the axioms of arithmetic it can be used to mean that nothing is known about zero divided by zero. Bottom can be added to any consistent, logical system without introducing any contradictions, so it allows division of zero by zero without contradiction. But what of division of a positive or negative number by zero? Denotational semantics has something to say about this, too, but it says it using the whole, complex, machinery of infinite sets. There is no doubt that one can describe transarithmetic in the language of denotational semantics, but the result is far more complex than transarithmetic, and far more mutable. Denotational semantics can be used to describe anything that can be programmed in a computer, but it does not, of itself, provide precise specifications of how to divide a number by zero.

Set theory, category theory

There are many other logical approaches to division by zero, and many ways of using set theory to talk about sets of results. These can all handle the absence of a numerical result, for zero divided by zero, by using the empty set of results with no elements, but that is quite different from having an element, nullity, that is zero divided by zero. In some of these approaches the empty set is used to denote zero so it would be very messy to use it all over again to represent nullity. It could be done, but the resulting theory would be far more complicated than transarithmetic.

Wheels

There is another, broad, approach to division by zero that was developed by Carlström (1973–). Wheels[9] are algebraic structures that allow division by an element zero. The generalisation of division is perfectly natural, from an algebraic point of view, but does not preserve the maximum possible information about magnitude and sign when applied to real numbers. Consequently, the arithmetic of wheels is quite different from transarithmetic. Again, there are many other algebraic approaches to division by zero, but none of them describes exactly the algebraic structure of transarithmetic.

NaN

There is also an international computer standard[12] that holds that zero divided by zero is a special object that describes a class of objects, each of which is *N*ot *a N*umber, *NaN*. This standard is ambiguous, but can be read in a consistent way so that no logical problems arise from the very odd property that *NaN* is not equal to itself. By contrast, nullity is equal to itself and, as a consequence, is much easier to use than *NaN*. I discuss *NaN* in the chapter *NaN*.

Readership

There have, no doubt, been many conscious and unconscious influences on my development of transarithmetic. And, just as I read historic Indian mathematicians from a modern perspective, so there must be many ways to read transarithmetic from the point of view of very sophisticated mathematical theories. My business, though, is to explain transarithmetic, and the transmathematics that develops from it, in a simple way so that it can be used by the widest possible range of people. As I infamously demonstrated, fifteen year old school pupils can learn to divide by zero in twenty minutes, so it will not take professional mathematicians very long to learn how to do it, though the general reader may be a little rusty on the school mathematics that is needed. Since my first demonstration, eleven year old children have learnt transarithmetic.

Presentation

I have tried to keep the presentation of mathematics very simple. When equations, or other formulas, appear in text, I punctuate them just like English. But when they appear as free standing text, as in [E 1.1], I do not punctuate them. As most English punctuation marks have a special mathematical meaning they are inherently ambiguous, forcing the reader to disambiguate them by context. This is easy to do in a block of text, because only short eye movements are required to check context, but freestanding equations are split off from blocks of text by some distance so that it takes long and, potentially, inaccurate eye movements to work out the context. This may not be a problem for fluent readers of mathematics, who may well prefer to have all equations punctuated, but it is confusing to the beginner and the dyslexic reader.

I have numbered equations, $E\ c.n$, by the letter, E, denoting an *E*quation, the *C*hapter number, c, and the *N*umber, n, of the equation within the chapter. Similarly, I label *A*xioms as $A\ c.n$, *D*efinitions as $D\ c.n$, and *T*heorems as $T\ c.n$ so that free standing formulas are clearly labelled by the role they perform in the development of a mathematical argument.

I also take care to give different things within a chapter individual names. For example, I do not refer to the topological space, T, and the topology, T, by relying on a change of font to make the distinction clear. Instead I talk of a topological space, S, and a topology, T. Nor do I follow the practice of using a single noun to refer to different kinds of things within a chapter. For example, I do not use “topology” to refer ambiguously to a topological space or a topology. While such contractions may be a normal part of the discourse of fluent mathematicians, they are nothing but a hindrance to the beginner and the dyslexic reader. In my view, such ambiguous uses should be expunged from all introductory texts. However, I do re-use names and symbols between chapters. To do otherwise would require a vast vocabulary which would, itself, be a hindrance to understanding.

In the on-line version of the book, text coloured read is a hyperlink to related material within the book or to external sources of information.

I have refrained from supplying an introductory chapter on mathematical notation. For many readers it would be completely unnecessary and for all readers it would delay the interesting story of how to divide by zero. Instead, I explain notations where they are first used, and summarise the use in the appendix, *Notation*. There is, therefore, an advantage to reading the chapters in sequence, but the reader should feel free to dip in and out of chapters, consulting the appendix whenever necessary. The reader who is a little rusty at mathematics might want to read just the chapters on *Pencil and Paper Methods*, elementary algebra, and the nature of numbers, before laying the book aside, on a coffee table, for the enjoyment of visitors.

Invitation

Everyone is welcome to send me a critique of this book, or to suggest a topic for a new chapter, or to offer a new chapter of their own. I am very happy to engage in discussions on transmathematics and to record upsets and progress here.

CHAPTER 2

Axioms of Transreal Arithmetic

Proofs

The arithmetic we use in our daily lives is a lot more detailed than many people suppose. Transreal arithmetic is defined in thirty-two axioms, given after the next section on notation. This is thirteen more than ordinary arithmetic. The extra axioms are all those that describe the properties of the *strictly transreal* numbers $-\infty, \infty, \Phi$. These are: [A4], [A5], [A9], [A10], [A11], [A15], [A16], [A20], [A21], [A22], [A23], [A24], [A25]. The remaining nineteen axioms are just the axioms of the ordinary arithmetic of real numbers, when the strictly transreal numbers are struck out from the transreal axioms. This can be verified by looking up the axioms of real arithmetic in a text book and showing the equivalence algebraically. It would be possible to give a formal proof that transreal arithmetic contains the whole of real arithmetic as a proper subset, but no one has given this proof so far.

It has been proved that the axioms of transreal arithmetic are self consistent. My colleague from Essex translated the axioms of transreal arithmetic into higher order logic and used a computer proof system to verify consistency.[6] This is the most detailed kind of proof it is possible to have, but it is not very readable. It would be possible to translate the computer proof into a human proof but, so far, no one has done so.

In fact, axiom [A32] is due to my colleague from Essex. He found a fault in my axiom that carried the transrational numbers into the transreal numbers. Axiom

[A7] is due to Andrew Adams (1969-), his axiom is far neater than my original version. Andrew found one redundant axiom in my original axiomatisation using a hand proof, and my colleague from Essex found four redundant axioms using machine proof. This demonstrates the advantage that machine proof has. It explores every detail of an axiomatisation and throws up all logical faults. Even so, a human mathematician, such as Andrew, can find a result before a computer does.

Machine proof has the huge advantage that it can be re-done very quickly, by the computer, when the user amends an axiom, but it requires a great deal of mathematical and programming knowledge to use machine proof effectively. I am very grateful to my colleague from Essex for the care he took in producing the proof.

Thirty two axioms is a small number, as machine proofs go, but it would be possible to reduce the number of axioms to one. A balance has to be struck, however, between laying out the axioms in a form that is useful to the human mathematician, and a form that is suitable for a computer. No doubt experience will suggest other axiomatisations but, for now, this is the only axiomatisation of the transreal numbers so we must make the best of it.

If the reader wants to show that transreal arithmetic is inconsistent then all that is needed is a proof that a contradiction follows from the axioms. The *Pencil and Paper Methods* are a very quick way to explore the axioms and to look for contradictions, but many people are tripped up by their ingrained habits of ordinary pencil and paper methods. It is better, therefore, to translate an attempted counter proof into an axiomatic proof. Having said that, I make very free use of the pencil and paper methods in the presentation of proofs. There is nothing wrong with using these methods, once it has been proved that they compute the same results as the axioms. Such a proof is given in the chapter on *Pencil and Paper Methods*. It would be possible to give a more detailed proof than is presented there but, so far, no one has produced such a proof.

Mathematicians say that ordinary, real numbers form an algebraic structure called a *field,* which can be axiomatised in just eight axioms. This is true, but real arithmetic also has ordering relationships so that, for example, we know that one is bigger than zero. Putting in axioms for ordering increases the number of axioms. It is possible to add some infinities to a field, but this requires yet more axioms. Finally, adding nullity, or the transreal infinities, breaks the field axioms so that transarithmetic is not a field. In fact, no name has been given to the algebraic structure of transreal numbers. All in all, thirty two axioms is not excessive for a total number system.

Notation

The strictly transreal numbers are: negative infinity $-\infty = (-1)/0$; positive infinity $\infty = 1/0$; and nullity $\Phi = 0/0$.

$+, -, \times, \div$ are, respectively, the operations of addition, subtraction, multiplication, and division. They apply to ordinary numbers in the ordinary way, but also apply to the strictly transreal numbers.

In ordinary mathematics, a^{-1}, is known as the *multiplicative inverse* and we have $a \times a^{-1} = \frac{a}{a} = 1$ when a is not zero. It comes as a shock to many people to discover that the multiplicative inverse is not the whole of division.

A more general form of division arises from the reciprocal: $\left(\frac{n}{d}\right)^{-1} = \frac{d}{n}$ In transarithmetic, the superscript minus one, -1, denotes the transreciprocal, as shown. This includes the ordinary reciprocal, which is defined via the multiplicative inverse. The *transreciprocal* also applies to the strictly transreal numbers which have no multiplicative inverse.

Parentheses, round brackets, are evaluated, as usual, from the innermost bracket to the outermost. The result is then written without brackets. For example, $(2 \times (4-3)) = (2 \times 1) = 2$, and $((2 \times 4) - 3) = (8-3) = 5$. Parentheses can be used to distinguish the negation of a single number from a subtraction of two numbers. Thus, $(2 \times (3-4)) = (2 \times (-1)) = -2$ and $3 + (-2) = 3 - 2 = 1$.

$=, \neq$ are the operations equals and not-equals.

$\neg a$ is true when a is false, and is false when a is true. The symbol "$\neg$" is known as "not." For example, $2 \neq 3$ means that two is not equal to three, which is a true statement, and $\neg(2 = 3)$ means the same thing, that two is not equal to three.

$a < b$, $a \leq b$, $a > b$, $a \geq b$ mean, respectively, a is less than b; a is less than or equal to b; a is greater than b; a is greater than or equal to b.

The comma, "," introduces an alternative. For example, $a \neq -\infty, \Phi$, means, "a is not equal to negative infinity and a is not equal to nullity."

$\pm a$ introduce the alternatives, $+a$ and $-a$.

The colon, ":" means "when" or "such that." For example, [A5], $a + \infty = \infty : a \neq -\infty, \Phi$, means, "$a$ plus infinity equals infinity, when a is not equal to negative infinity and a is not equal to nullity."

$a \Rightarrow b$ means that if a is true then b is true. It is also read as, "a implies b."

$a \Leftrightarrow b$ means, "if a is true then b is true and if b is true then a is true". This is also read as, "a is true if and only if b is true."

$\exists a$ means, "there exists an a."

$\forall a$ means, "for all a."

$a \wedge b$ is true when both of a, b are true, and is false when either or both of a, b are false. This is read as "a and b."

$a \vee b$ is true when either or both of a, b are true, and is false when both of a, b are false. This is read as "a or b."

The function $\mathrm{sgn}(a)$ is used as a shorthand so that $\mathrm{sgn}(a) = -1$ when $a < 0$, $\mathrm{sgn}(a) = 0$ when $a = 0$, $\mathrm{sgn}(a) = 1$ when $a > 0$, and $\mathrm{sgn}(a) = \Phi$ when $a = \Phi$.

Axioms

Axioms are expressions which are assumed to be true without proof. Logical inference rules, that is to say, proof methods, are applied to axioms to obtain derived theorems. The axioms and derived theorems together comprise the whole set of theorems.

Additive Associativity	$a+(b+c) = (a+b)+c$	[A1]
Additive Commutativity	$a+b = b+a$	[A2]
Additive Identity	$0+a = a$	[A3]
Additive Nullity	$\Phi+a = \Phi$	[A4]
Additive Infinity	$a+\infty = \infty : a \neq -\infty, \Phi$	[A5]
Subtraction as Sum with Opposite	$a-b = a+(-b)$	[A6]
Bijectivity of Opposite	$-(-a) = a$	[A7]
Additive Inverse	$a-a = 0 : a \neq \pm\infty, \Phi$	[A8]
Opposite of Nullity	$-\Phi = \Phi$	[A9]
Non-null Subtraction of Infinity	$a-\infty = -\infty : a \neq \infty, \Phi$	[A10]
Subtraction of Infinity from Infinity	$\infty-\infty = \Phi$	[A11]
Multiplicative Associativity	$a \times (b \times c) = (a \times b) \times c$	[A12]
Multiplicative Commutativity	$a \times b = b \times a$	[A13]
Multiplicative Identity	$1 \times a = a$	[A14]
Multiplicative Nullity	$\Phi \times a = \Phi$	[A15]
Infinity Times Zero	$\infty \times 0 = \Phi$	[A16]
Division	$a \div b = a \times (b^{-1})$	[A17]

Multiplicative Inverse $a \div a = 1 : a \neq 0, \pm\infty, \Phi$ [A18]

Bijectivity of Reciprocal $(a^{-1})^{-1} = a : a \neq -\infty$ [A19]

Reciprocal of Zero $0^{-1} = \infty$ [A20]

Reciprocal of the Opposite of Infinity $(-\infty)^{-1} = 0$ [A21]

Reciprocal of Nullity $\Phi^{-1} = \Phi$ [A22]

Positive $\infty \times a = \infty \Leftrightarrow a > 0$ [A23]

Negative $\infty \times a = -\infty \Leftrightarrow 0 > a$ [A24]

Positive Infinity $\infty > 0$ [A25]

Ordering $a - b > 0 \Leftrightarrow a > b$ [A26]

Less Than $a > b \Leftrightarrow b < a$ [A27]

Greater Than or Equal $a \geq b \Leftrightarrow (a > b) \vee (a = b)$ [A28]

Less Than or Equal $a \leq b \Leftrightarrow b \geq a$ [A29]

Quadrachotomy

Exactly one of: $(a < 0)$, $(a = 0)$, $(a > 0)$, $(a = \Phi)$

[A30]

Distributivity

$a \times (b + c) = (a \times b) + (a \times c) : \neg((a = \pm\infty) \wedge (\mathrm{sgn}(b) \neq \mathrm{sgn}(c)) \wedge (b + c \neq 0, \Phi))$

[A31]

Lattice Completeness

The set, X, of all transreal numbers, excluding Φ, is lattice complete because

$\forall Y : Y \subseteq X \Rightarrow (\exists u \in X : (\forall y \in Y : y \leq u) \wedge (\forall v \in X : (\forall y \in Y : y \leq v) \Rightarrow u \leq v))$

[A32]

Notes

Axiom [A17] uses a superscripted minus one, b^{-1}. This superscript has several uses in ordinary mathematics. It can mean the generalised inverse, the inverse, the multiplicative inverse, or the reciprocal. In ordinary mathematics the multiplicative inverse and the reciprocal are synonyms for each other, but all of the inverses are different from each other. In transreal arithmetic the multiplicative inverse and the reciprocal are different things, and the superscript minus one means the reciprocal not the multiplicative inverse. As [A20], [A21], [A22] show, transreal arithmetic has more reciprocals than real arithmetic does. Furthermore, [A19] shows that the reciprocal of the reciprocal of a number does not return us to the number in the single case that the number is minus infinity. This single non-bijectivity marks an important difference with real arithmetic.

Axiom [A23] defines all positive numbers in terms of infinity. By contrast, most other axiomatisations of arithmetic define a special class of positive numbers by giving a complicated construction and then slowly building up to negative numbers. We get negative numbers in the very next axiom, [A24].

Axiom [A25] says that infinity is greater than zero. Consequently we can write just ∞ for $+\infty$ so that positive infinity is written without a sign, following the usual convention for positive numbers. The axiom is important though. Knowing that infinity is greater than zero is enough to prove that there is no transreal number bigger than infinity. By contrast, ordinary arithmetic cannot prove that infinity is big, this has to be given as an extra axiom in ordinary arithmetic.

Ordinary arithmetic has an axiom of trichotomy which means that a number is less than zero, equal to zero, or greater than zero. But [A30] adds a fourth case, a number can be equal to nullity. This axiom means that nullity is incomparable with any number marked out on a scale of numbers which may run anywhere from minus infinity to infinity. This forces nullity to have certain topological properties that are disjoint from the extended, real-number line. It also raises interesting issues about the nature of nullity. On a practical level, it means that many mathematical proofs that exploit trichotomy now have to consider a fourth case where a number is nullity. Very often these proofs survive, but sometimes they are overturned. This can be accommodated by extending a theorem so that it deals explicitly with nullity.

Axiom [A31] shows that transreal arithmetic is non-distributive in certain, very restricted cases. The consequence of this is that formulas in transarithmetic are very sensitive to the order in which any sub-expressions are written down or bracketed together. It comes as a shock to many people that ordinary arithmetic is sensitive to such orderings (for example, when division is re-ordered as a multiplication to avoid the possibility of division by zero, as discussed in the chapter on the See Saw), but there are many more cases to deal with in transarithmetic. Typically, proofs in transarithmetic are longer than ordinary proofs, because they have to deal with these conditions, but the extra labour is illusory. No amount of work with ordinary arithmetic can deal with division by zero so conventional mathematics must adopt more advanced theory to get any answers in these cases. This theory is invariably much harder to work with than arithmetic so the apparent extra labour of transarithmetical proofs is actually less effort than would have to be used on a conventional mathematical proof. And now, a health warning. Computer algebra systems seldom check side conditions, such as division by zero, so they often give false answers. It would be possible to do far more checking, using transreal arithmetic, but, so far, no one has produced a computer algebra system using transreal arithmetic.

Axiom [A32] has the effect of carrying transrational numbers into transreal numbers. The formula has too much detail, too heavily nested in relative clauses, to be meaningful to a human reader on its own. It is not until these details are clumped together in a picture, or in the words, *every subset of X has a supremum,* that we can comprehend its meaning. Providing, of course, that we know what a supremum is and how it is used.[15]

To my mind, the last axiom is the closest I have ever come to reading a sentence of Martian. But we have the axioms now and using them is an awful lot simpler than working them out in the first place.

Axiomatic proofs

In an axiomatic system, a proof starts from some axioms and makes a sequence of moves that are justified by logical inference rules. The axioms and the expressions which result from each move are considered to be theorems. This often produces an awful lot of very dull theorems on the way to an interesting one. This process can be seen in the paper on the proof of the consistency of transreal arithmetic. There are very many theorems listed in that paper,[6] but scarcely half of them are interesting, to my mind.

Axiomatic proofs are very detailed. Ideally they are written in nothing but mathematical symbols, but this makes them very difficult to read. Nonetheless, we can illustrate how a computer goes about finding an axiomatic proof. We will look at two proofs: that every transreal number can be divided by zero, and that there is no number bigger than infinity. These results are useful in themselves, and it is interesting that it is impossible to obtain these proofs in ordinary arithmetic, but very easy in transreal arithmetic.

Machine-like proof that every transreal number can be divided by zero

The transreal numbers are defined by their axioms so we want to prove that there is an axiom, or a chain of moves starting from some axioms, which proves that there is no restriction on which numbers can be divided by zero.

We scan down the list of axioms looking for the division symbol, $\div$. We find it in axiom [A17]. This tells us that $a \div b = a \times (b^{-1})$. This means that the division of any transreal numbers a, b is defined in terms of the transreciprocal, b^{-1}. This will meet the goal of proving that all transreal numbers can be divided by zero, providing that we can prove the sub-goal the every number, b, has a transreciprocal, b^{-1}. We now scan down the list of axioms looking for the superscript minus one. We find it in axiom [A19], $(a^{-1})^{-1} = a : a \neq -\infty$. This tells us that the transreciprocal of the transreciprocal of a number is the number itself, unless the number is negative infinity. This tells us nothing about the existence of the transreciprocal, b^{-1}, so we give up on [A19] temporarily and carry on the search for a superscript minus one. We find it at [A20], $0^{-1} = \infty$. This tells us that the transreciprocal of zero is infinity. We make a note of this and carry on the search for a superscript minus one. We find it at [A21], $(-\infty)^{-1} = 0$. This tells us that the transreciprocal of negative infinity is zero. We make a note of this and carry on the search for a superscript minus one. We find it at [A22], $\Phi^{-1} = \Phi$. This tells us that the transreciprocal of nullity is nullity. We make a note of this and carry on the search for a superscript minus one. There is no further superscript minus one. We check our notes, we have proved that $b = 0, -\infty, \Phi$ have a transreciprocal b^{-1}, but we have not proved that all transreal numbers have a transreciprocal. We temporarily give up on the sub-goal of proving that every number, b, has a transreciprocal, b^{-1}, and carry on our search for the symbol, $\div$. We find it at [A18], $a \div a = 1 : a \neq 0, \pm\infty, \Phi$. This tells us that every number, a, can be divided by itself, unless the number is zero, infinity, negative infinity, or nullity. This does not appear to tell us anything about the transreciprocal, but we substitute [A17], $a \div b = a \times (b^{-1})$, into [A18], $a \div a = 1 : a \neq 0, \pm\infty, \Phi$, to give $a \times (a^{-1}) = 1 : a \neq 0, \pm\infty, \Phi$. This tells us that every a has a transreciprocal a^{-1}, except when a is zero, infinity, negative infinity, and nullity. We check our notes, we have already proved that every a has a transreciprocal, a^{-1}, when a is zero, negative infinity, and nullity. This just leaves the case a is infinity to prove. We temporarily give up on [A18] and try to satisfy the sub goal ∞^{-1} exists. We cannot find any axiom with ∞^{-1} in it so we search for a way of re-writing infinity. We find that [A20], $0^{-1} = \infty$, allows us to re-write infinity as the transreciprocal of zero. Now we search for axioms that tell us something about the transreciprocal of zero. We find that [A17], $a \div b = a \times (b^{-1})$, tells us that $a \div 0 = a \times (0^{-1})$, but this does not tell us what 0^{-1} is. We temporarily give up on [A17] and continue the search for axioms that tell us something about the transreciprocal of zero. We find that [A19],

$(a^{-1})^{-1} = a : a \neq -\infty$, tells us that $(0^{-1})^{-1} = 0$, so now we know that $\infty^{-1} = (0^{-1})^{-1} = 0$. We check our notes and we find that we have proved that every number can be divided by zero. We declare success, and throw away all of our temporary notes and all of the records of the sub-goals we had temporarily given up on.

Now, I have skimmed over a lot of detail on how a computer comes up with a proof, but this gives a flavour of it. The search for goals is very mechanical, but is helped along by rules of inference that allow the machine to substitute one symbol for another. The proofs are very detailed, very long, follow blind alleys in excruciating detail, and seem to declare success on the achievement of the most trivial result. This seems very messy to the human mind. What the human reader wants to see is a neater presentation of the proof. We can get this by pouring over the machine proof. In presenting the human proof we ignore boring detail, and use mathematical jargon to avoid long strings of mathematical symbols. Here is a re-write of the proof.

Human proof that every transreal number can be divided by zero

Proof that every transreal number can be divided by zero. Every transreal number can be divided by zero if every transreal number has a transreciprocal, [A17]. Every transreal number, other than $0, \pm\infty, \Phi$, has a transreciprocal via [A18]. All of these numbers, except ∞, has a transreciprocal given by, [A20], [A21], or [A22]. Now $\infty^{-1} = (0^{-1})^{-1} = 0$ via [A19] and [A20] so that ∞ has a transreciprocal. Thus, every transreal number has a transreciprocal, and this completes the proof.

Comparison of machine and human proofs

The machine-like proof is much longer than the human proof. Typically a machine proof will be forty times longer than the corresponding human proof so a mathematician-programmer has to do forty times as much writing as a human mathematician. This makes the development of machine proofs very slow, but it has the advantage that the computer follows every single detail of the proof, whereas human mathematicians are notorious for missing out detail. It also means that if an axiom is amended by the mathematician-programmer then the computer can work through the details very quickly. It took my colleague, from Essex, several weeks to set up the axioms of transreal arithmetic in a computer, and do all of the other things needed to prepare a computer proof, but it took the computer only a few minutes to prove the consistency of the axioms. If my colleague, or another mathematician, were to re-write the computer proof, as a human proof, it would take many weeks to boil down the detail into a description that is accessible to the human reader. Someone might do this one day but, so far, no one has.

Human proof that there is no transreal number bigger than infinity

We wish to prove that there is no transreal number bigger than infinity. If there is some number $a > \infty$ then $a - \infty > 0$ by [A26]. But if a is real then $a - \infty = -\infty < 0$ by [A10], [A24] so that this condition is not satisfied. Alternatively, if $a = \infty, \Phi$ then $a - \infty = \Phi$ via [A4], [A11] so that, again, the condition is not satisfied. Finally, if $a = -\infty$ then $a - \infty = -\infty < 0$ via [A10], [A24] and the condition is not satisfied. This exhausts all alternatives, and completes the proof.

Non-triviality of the transreal axioms

Ordinary arithmetic obeys the axioms of a field, amongst other axioms, but it can be shown that there is a trivial solution to the field axioms when a field has a single element, e. In this case $e = 0 = 1$. By contrast, there is no trivial solution to the transreal axioms.[6] Any system that obeys the transreal axioms has six distinct elements, $\Phi, -\infty, -1, 0, 1, \infty$.

Looking forward

We will see shortly how to carry out transarithmetic using the arithmetic of fractions. This will make proofs much simpler and will allow us to present more detailed proofs in a smaller space than can be done with axiomatic methods.

CHAPTER 3

Pencil and Paper Methods

Introduction

If you are a primary school teacher then you already teach your pupils how to divide by zero. This chapter will help you to clarify the message you give your pupils and will give you a conception of infinity and nullity that you can pass on to them. The chapter contains more than is needed in primary school. If you want to teach division by zero, you will have to select parts of the material, re-order it to fit your syllabus, and work with your colleagues to deliver an appropriate syllabus across the primary years. There is very little you can do on your own unless, of course, you are a sole teacher in a village school.

If you are a mathematics teacher in a secondary school then you already teach your pupils how to divide by zero. If you want to teach division by zero explicitly then you will need more material than is given in this chapter. In time, I might get round to extending all of the mathematics taught in secondary schools, but I will not live long enough to extend all of the mathematics that appears in the mathematical tables your pupils use. These tables summarise millennia of mathematical development. I cannot hope to survey and extend it all. If you teach middle or late years in secondary school then you can teach all of the material in this chapter to your own class, without reference to earlier or later years.

When you teach mathematics you might have one of several objects in mind. You might want to teach conventional mathematics without regard to transreal mathematics. This is the most conservative position you can adopt. Alternatively, you might want to teach conventional mathematics in a way which does not obstruct the learning of transreal mathematics. This will require you to be a little more careful in your teaching, but will give no hint to your pupils that it is both possible and useful to divide by zero. Finally, you might want to teach division by zero explicitly. This will require you to exercise your professional judgement to overturn advice in your current syllabus. Do this only if you are confident of your position. I accept no legal responsibility for your actions. Indeed, English law protects me in challenging received wisdom so I can act with a reasonable assurance of impunity. You might not be so lucky. (Interestingly, Scottish law provides stronger protection. A protection which is needed by those who engage in paradigm breaking research. To this extent, at least, Scottish law encourages science more strongly than English law. But let's stay friends, and have no recourse to the law!)

I can help teachers and educationalists by putting you in touch with each other, and by helping you to develop teaching materials. I was once a research psychologist and can help you carry out research on teaching methods. I am currently a university lecturer in Computer Science and can make presentations or lead workshops at your professional meetings. I have experience of professional and political lobbying, and can help you to win arguments about the teaching of mathematics and computing. I am a resource you can call on to assess transreal mathematics and to explore its teaching.

If you are a pupil you can teach yourself transreal mathematics. First, you need to be a *G*enius. You must be smarter than your class mates so that you can learn all of the mathematics they learn in school, and still have smarts left over to extend that mathematics to cope with division by zero. Second, you must have *E*nthusiasm. You must keep on learning, even when there is no one there to support you. Use the web, and use your local library. You can find interesting topics on the web, but you need text books to get detail. Third, achieve *E*xcellence. If you work as hard as you will have to, to understand division by zero, then you will achieve excellence in your school work and, perhaps, you will achieve excellence as a mathematician. Finally, you will be rewarded with *K*udos from your school friends and, later, your professional colleagues. Put this all together – *G*enius, *E*nthusiasm, *E*xcellence, and *K*udos – and you know what it is to be a true Maths Geek. When your friends make fun of you for taking learning seriously, you will know better than them what learning means.

Transreal number line

Now, learning to divide by zero is actually very easy. First, you need to keep a picture in mind of how all the transreal numbers fit together.

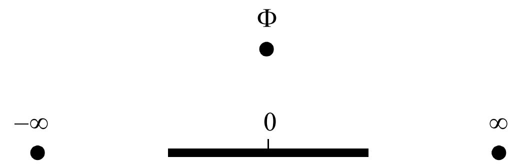

Figure 3.1: The transreal number line

All of the numbers taught in primary schools lie on the thick black line. The counting numbers (1, 2, 3, …) lie on the line to the right of zero. Fractions also lie on the line. Many fractions are in the gaps between the counting numbers, but the counting numbers are also fractions. In some countries, primary schools teach negative numbers. These lie on the line to the left of zero.

All of the numbers taught up to, at least, the middle years of secondary school lie on the line. These are called *real* numbers and the line is called the *real-number line*. Some countries teach infinity as an unboundedly large real number. This is a much smaller kind of infinity than transreal infinity. Transreal infinity is so big that it breaks off from the real-number line and occurs after a gap. Transreal infinity is the biggest infinity there is. All other infinities, that can be drawn on this diagram, lie in the gap or asymptotically far to the right-hand edge of the real-number line. Any fixed real number is called a *finite* number, only variables can move on a number line. There is also a negative infinity. Some countries teach complex numbers. If you want to teach transcomplex numbers then you will have to wait until details on these numbers are published.

No country teaches nullity in primary or secondary school. Nullity lies off the real-number line extended up to the infinities.

Nullity

The fraction $\frac{0}{0}$ has a name. It is called *nullity.* It also has a symbol, Φ . This is the symbol, capital Phi, from the Greek alphabet. It is written as a Roman capital, *I,* with a ring drawn on top of the middle part of the *I.* The ancient Greeks and Romans did not know about the number nullity. Nullity was invented in 1997.[4] It is a *non-finite* number. Now that nullity has been invented, we can write $\Phi = \frac{0}{0}$.

Infinity

The fraction $\frac{1}{0}$ has a name. It is called *infinity.* It also has a symbol, ∞ . This is drawn as a figure eight on its side. The name infinity is shared by many different mathematical objects. When we use transreal infinity we write $\infty = \frac{1}{0}$. Transreal infinity is equal to any positive transreal-number divided by zero. For example:

$$\infty = \frac{1}{0} = \frac{2}{0} = \frac{3}{0} = \frac{\sqrt{2}}{0} = \frac{\pi}{0} = \frac{\infty}{0}$$

Infinity is also a *non-finite* number.

Negative infinity

The fraction $\frac{-1}{0}$ has a name. It is called *negative infinity* or *minus infinity.* It also has a symbol, $-\infty$. When we use transreal, negative infinity we write $-\infty = \frac{-1}{0}$. Transreal, negative infinity is equal to any negative number divided by zero. For example:

$$-\infty = \frac{-1}{0} = \frac{-2}{0} = \frac{-3}{0} = \frac{-\sqrt{2}}{0} = \frac{-\pi}{0} = \frac{-\infty}{0}$$

Negative infinity is also a *non-finite* number.

Sign

All countries agree that some numbers can be *positive* or else *negative* and distinguish these with the signs, +, and, -, respectively. But countries differ on how zero is treated. Some countries say that zero is positive, others say that it is neither positive nor negative, it is just *zero.* English speaking countries say that zero is neither positive nor negative. That is how I use zero. You already know how your culture treats zero and how it is taught in schools. If you live in a non-English speaking country, you might have to be extra careful when translating mathematical publications in English for use at home, but you already know this. The number nullity is not negative, it is not zero, and it is not positive. Nullity has the sign *nullity* just as, in English speaking countries, zero has the sign *zero.*

Measuring and weighing

Primary schools teach pupils how to measure things with a ruler, a container, and a weighing scale. All of these measurements are finite. Infinity is bigger than anything anyone has ever measured. Even if you add a kilo of chocolate to a weighing scale, every minute of the day, for ever and ever, this is still less than a transreal infinity of chocolate. If you could add an infinite weight to one pan of a weighing scale, it would move the scales so as to outweigh anything else you could put in the other pan.

Nullity has never been measured. Even if you could put nullity in one pan of a weighing scale it would not change the position of the weighing scale no matter what else is in the other pan. Nullity does not register a change on any measuring device.

Despite the fact that no one has ever measured a non-finite quantity, such as infinity or nullity, non-finite quantities can be calculated in mathematical formulas.

Secondary schools teach pupils how to measure voltages that may be positive or negative, but are always finite. Negative infinity is less than any finite number and is less than positive infinity.

Nullity is equal to itself, but is not less than, equal to, or greater than, any other number.

Transreal fraction

A *transreal fraction* is a number, $\frac{n}{d}$, where n and d are any real numbers. Here n is called the *numerator* of the fraction and d is called the *denominator* of the fraction. For example, these are all transreal fractions:

$$\frac{0}{0}, \frac{0}{1}, \frac{0}{-1}, \frac{1}{0}, \frac{-1}{0}, \frac{1}{1}, \frac{-1}{1}, \frac{1}{-1}, \frac{-1}{-1}, \frac{\pi}{-\sqrt{2}}$$

In some countries, primary schools teach that the numerator and denominator of a fraction must be integers. This is a more restricted view of fractions than I use.

Proper transreal-fraction

A *proper transreal-fraction* is a transreal fraction with a non-negative denominator. For example, these are all proper transreal-fractions:

$$\frac{0}{0}, \frac{0}{1}, \frac{1}{0}, \frac{-1}{0}, \frac{1}{1}, \frac{-1}{1}, \frac{-\pi}{\sqrt{2}}$$

An improper transreal-fraction, i, is converted to a proper transreal-fraction, f, by negating both the numerator and the denominator:

$$\text{Let } i = \frac{n}{-d} \text{ with } -d < 0 \text{ then}$$

$$f = \frac{n}{-d} = \frac{-n}{-(-d)} = \frac{-n}{d}$$

Here is a numerical example of converting the improper transreal-fraction $\frac{1}{-2}$ into the proper transreal-fraction $\frac{-1}{2}$.

$$\frac{1}{-2} = \frac{-1}{-(-2)} = \frac{-1}{2}$$

In my view, it is simpler to negate the numerator and denominator than to multiply both by minus one, but I do not know which approach leads to better learning in the many countries of the world. Similarly, I think it is easiest to teach transreal multiplication, division, addition, and subtraction via proper transreal-fractions, but other approaches are possible. Again, I do not know which approach works well in any specific country. Bear in mind, too, that all of my professional experience of teaching is in tertiary education. I cannot advise primary and secondary teachers on how to teach. You must use your own professional judgement.

Multiplication

Two proper transreal-fractions are multiplied like this:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

For example:

$$\infty \times 3 = \frac{1}{0} \times \frac{3}{1} = \frac{1 \times 3}{0 \times 1} = \frac{3}{0} = \infty$$

If you want to teach ordinary mathematics then teach your pupils the above rule, but tell them that $b, d \neq 0$. If you want to teach ordinary mathematics without obstructing the learning of transreal mathematics then teach the above rule, but do not give any examples involving zero denominators. If you want to teach division by zero explicitly then teach the above rule and do give examples involving zero denominators.

Division

Two proper transreal-fractions are divided like this:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

For example:

$$\infty \div -3 = \frac{1}{0} \div \frac{-3}{1} = \frac{1}{0} \times \frac{1}{-3} = \frac{1}{0} \times \frac{-1}{-(-3)} = \frac{1}{0} \times \frac{-1}{3} = \frac{1 \times -1}{0 \times 3} = \frac{-1}{0} = -\infty$$

If you want to teach ordinary mathematics then teach your pupils the above rule, but tell them that $b, c, d \neq 0$. If you want to teach ordinary mathematics without obstructing the learning of transreal mathematics then teach the above rule, but do not give any examples involving zero denominators. If you want to teach division by zero explicitly then teach the above rule and do give examples involving zero denominators.

Addition

There are two rules for adding proper transreal-fractions. The first is a special rule for adding two infinities, the second is a general rule that applies in every other case.

$$\text{Special rule: } (\pm\infty) + (\pm\infty) = \frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0}$$

$$\text{General rule: } \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

For example:

$$\infty + \infty = \frac{1}{0} + \frac{1}{0} = \frac{1+1}{0} = \frac{2}{0} = \infty \text{ using the special rule}$$

$$\infty + \frac{2}{3} = \frac{1}{0} + \frac{2}{3} = \frac{1 \times 3 + 0 \times 2}{0 \times 3} = \frac{3+0}{0} = \frac{3}{0} = \frac{1}{0} = \infty \text{ using the general rule}$$

If you want to teach ordinary mathematics then teach your pupils the above general rule, but tell them that $b, d \neq 0$. And teach them the following rule for adding numbers with a common denominator, where $c \neq 0$.

$$\text{Common denominator rule: } \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

If you want to teach ordinary mathematics without obstructing the learning of transreal mathematics then teach the above general rule, but do not give any examples involving zero denominators, and do not teach the common denominator rule. If you want to teach division by zero explicitly then teach the above special and general rule, but not the common denominator rule, and do give examples involving zero denominators.

Subtraction

Two proper transreal-fractions are subtracted like this:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$$

For example:

$$\infty - \infty = \frac{1}{0} - \frac{1}{0} = \frac{1}{0} + \frac{-1}{0} = \frac{1+(-1)}{0} = \frac{1-1}{0} = \frac{0}{0} = \Phi$$

If you want to teach ordinary mathematics then teach your pupils the above rule, but tell them that $b, d \neq 0$. If you want to teach ordinary mathematics without obstructing the learning of transreal mathematics then teach the above rule, but do not give any examples involving zero denominators. If you want to teach division by zero explicitly then teach the above rule and do give examples involving zero denominators.

School syllabus

The material above is just about all the transreal arithmetic that is needed in primary school, and is too much for some countries, though if you are a primary school teacher you might want to develop more activities involving infinity and nullity. If you are a secondary school teacher you will want to draw on more sophisticated examples of transmathematics as are given, or cited, elsewhere in this book. In either case, you might want to know why the above methods work. The following proofs explain this.

Proof of consistency with real arithmetic

All of the methods presented above are ordinary arithmetical algorithms, as taught in schools, except that they allow division by zero. If all of the denominators happen to be non-zero then the methods compute all and only the results of ordinary, real-numbered, arithmetic. If a denominator is zero then an immediate result of $-\infty$, ∞, or Φ is produced, but these non-finite numbers are not real numbers so there is no overlap with the real numbers and no possibility of producing a contradictory, real-numbered, immediate result. If a denominator is Φ then the result is Φ. Again, there is no overlap with the real numbers and no possibility of producing a contradictory, real-numbered, immediate result. Finally, if a denominator is $-\infty$ or ∞ then the immediate result is 0 or Φ. A result of 0 could generate a contradictory real-numbered result, but $-\infty$ and ∞ are non-finite numbers so they cannot occur in a real-numbered calculation. We have now surveyed all of the possible ways in which non-finite numbers can enter a calculation and have found that none of them generate contradictions with the real numbers.

This proof does what is wanted, it shows that the methods presented are consistent with ordinary, real arithmetic; but it reads more like an alibi than a proof. Still, an alibi is good enough. But we want to know more, we want to know that the methods themselves are consistent even when they involve non-finite numbers. Proving this from first principles would be a very great deal of work, but help is at hand. It has already been proved that transreal arithmetic is consistent[6] so if we can show that the methods support the same operations as the axioms then we will know that the methods are consistent. More to the point, we will know that the methods implement transreal arithmetic. Whenever we want to compute a transreal result we will then have the choice of using a laborious method of axiomatic proof or a quick method of pencil and paper calculation. A big advantage of the pencil and paper methods is that they can be implemented in the hardware of a computer chip so that computers can provide us with very quick transreal computation.

Proof that the pencil and paper methods implement all of the axioms of transreal arithmetic

We want to know that the pencil and paper methods implement the axioms of transreal arithmetic, as set out in the chapter *Axioms of Transreal Arithmetic*. In doing this, we assume that the methods implement real arithmetic. Consequently, we do not need to establish *Lattice Completeness* [A32] whose only role is to carry the axioms of transrational arithmetic over to transreal arithmetic. Nor do we need to establish *Additive Inverse* [A8] or *Multiplicative Inverse* [A18], because these are just axioms of real arithmetic. Nor do we need to establish *Positive Infinity* [A25], because this is just a syntactic definition which specifies the sign of the infinity symbol. Nor do we need to establish *Subtraction as Sum with Opposite* [A6], *Division* [A17], *Ordering* [A26], *Less Than* [A27], *Greater Than or Equal* [A28], or *Less Than or Equal* [A29], because these are just syntactic definitions that define symbols in terms of symbols already defined by other axioms. Indeed, this is a good reason to omit some or all of these axioms from an axiomatisation of the transreal numbers. We now demonstrate that the methods implement the remaining axioms. We starting with the easy demonstrations and use these to obtain the harder demonstrations.

Additive identity

The *Additive Identity* axiom [A3] says $0 + a = a$ for all transreal a. Now $0 + a = \frac{0}{1} + \frac{n}{d}$ for some real n, d such that $a = \frac{n}{d}$. It is a theorem of transreal arithmetic, see [T20] in,[6] that $0 \neq \infty$ so we use the general rule for addition, which gives us: $\frac{0}{1} + \frac{n}{d} = \frac{0 \times d + 1 \times n}{1 \times d} = \frac{n}{d} = a$. This establishes that the methods implement axiom [A3] for all transreal a.

Having demonstrated, once, how to make an appeal to real arithmetic to establish transreal results, we can be less fussy in presenting the following proofs.

Additive nullity

The *Additive Nullity* axiom [A4] says $\Phi + a = \Phi$ for all transreal a. The methods implement this axiom as follows.

$$\Phi + a = \frac{0}{0} + \frac{n}{d} = \frac{0 \times d + n \times 0}{0 \times 0} = \frac{0}{0} = \Phi$$

Bijectivity of opposite

The *Bijectivity of Opposite* axiom [A7] says $-(-a) = a$ for all transreal a. The methods implement this axiom as follows.

$$-(-a) = -\left(-\frac{n}{d}\right) = \frac{-(-n)}{d} = \frac{n}{d} = a$$

Opposite of nullity

The *Opposite of Nullity* axiom [A9] says $-\Phi = \Phi$. The methods implement this axiom as follows.

$$-\Phi = -\frac{0}{0} = \frac{-0}{0} = \frac{0}{0} = \Phi$$

Subtraction of infinity from infinity

The *Subtraction of Infinity from Infinity* axiom [A11] says $\infty - \infty = \Phi$. The methods implement this axiom as follows.

$$\infty - \infty = \frac{1}{0} - \frac{1}{0} = \frac{1}{0} + \frac{-1}{0} = \frac{1 + (-1)}{0} = \frac{0}{0} = \Phi$$

Multiplicative commutativity

The *Multiplicative Commutativity* axiom [A13] says $a \times b = b \times a$ for all transreal a, b. The methods implement this axiom as follows.

$$a \times b = \frac{n_a}{d_a} \times \frac{n_b}{d_b} = \frac{n_a \times n_b}{d_a \times d_b} = \frac{n_b \times n_a}{d_b \times d_a} = \frac{n_b}{d_b} \times \frac{n_a}{d_a} = b \times a$$

Multiplicative identity

The *Multiplicative Identity* axiom [A14] says $1 \times a = a$ for all transreal a. The methods implement this axiom as follows.

$$1 \times a = \frac{1}{1} \times \frac{n}{d} = \frac{1 \times n}{1 \times d} = \frac{n}{d} = a$$

Multiplicative nullity

The *Multiplicative Nullity* axiom [A15] says $\Phi \times a = \Phi$ for all transreal a. The methods implement this axiom as follows.

$$\Phi \times a = \frac{0}{0} \times \frac{n}{d} = \frac{0 \times n}{0 \times d} = \frac{0}{0} = \Phi$$

Infinity times zero

The *Infinity Time Zero* axiom [A16] says $\infty \times 0 = \Phi$. The methods implement this axiom as follows.

$$\infty \times 0 = \frac{1}{0} \times \frac{0}{1} = \frac{1 \times 0}{0 \times 1} = \frac{0}{0} = \Phi$$

Reciprocal of zero

The *Reciprocal of Zero* axiom [A20] says $0^{-1} = \infty$. The methods implement this axiom as follows.

$$0^{-1} = 1 \div 0 = \frac{1}{1} \div \frac{0}{1} = \frac{1}{1} \times \frac{1}{0} = \frac{1 \times 1}{1 \times 0} = \frac{1}{0} = \infty$$

Reciprocal of the opposite of infinity

The *Reciprocal of the Opposite of Infinity* axiom [A21] says $(-\infty)^{-1} = 0$. The methods implement this axiom as follows.

$$(-\infty)^{-1} = 1 \div -\infty = \frac{1}{1} \div \frac{-1}{0} = \frac{1}{1} \times \frac{0}{-1} = \frac{1}{1} \times \frac{-0}{-(-1)} = \frac{1}{1} \times \frac{0}{1} = \frac{1 \times 0}{1 \times 1} = \frac{0}{1} = 0$$

Reciprocal of nullity

The *Reciprocal of Nullity* axiom [A22] says $\Phi^{-1} = \Phi$. The methods implement this axiom as follows.

$$\Phi^{-1} = 1 \div \Phi = \frac{1}{1} \div \frac{0}{0} = \frac{1}{1} \times \frac{0}{0} = \frac{1 \times 0}{1 \times 0} = \frac{0}{0} = \Phi$$

Multiplicative associativity

The *Multiplicative Associativity* axiom [A12] says $a \times (b \times c) = (a \times b) \times c$ for all transreal a, b, c. The methods implement this axiom as follows.

$$a \times (b \times c) = \frac{n_a}{d_a} \times \left(\frac{n_b}{d_b} \times \frac{n_c}{d_c}\right) = \frac{n_a}{d_a} \times \frac{n_b \times n_c}{d_b \times d_c} = \frac{n_a \times n_b \times n_c}{d_b \times d_b \times d_c} = \frac{n_a \times n_b}{d_b \times d_b} \times \frac{n_c}{d_c}$$
$$= \left(\frac{n_a}{d_a} \times \frac{n_b}{d_b}\right) \times \frac{n_c}{d_c} = (a \times b) \times c$$

Additive commutativity

The *Additive Commutativity* axiom [A2] says $a + b = b + a$ for all transreal a, b, c.

Using the special addition method we have:

$$a + b = \frac{n_a}{0} + \frac{n_b}{0} = \frac{n_a + n_b}{0} = \frac{n_b + n_a}{0} = \frac{n_b}{0} + \frac{n_a}{0} = b + a$$

Using the general addition method we have:

$$a + b = \frac{n_a}{d_a} + \frac{n_b}{d_b} = \frac{n_a \times d_b + d_a \times n_b}{d_a \times d_b} = \frac{n_b \times d_a + d_b \times n_a}{d_b \times d_a} = b + a$$

This exhausts the addition methods which apply and, as both approaches confirm the axiom, the axiom is implemented by the methods.

Having demonstrated, once, how to perform a census of applicable methods, we can be less fussy in presenting the following proofs.

Additive infinity

The *Additive Infinity* axiom [A5] says $a + \infty = \infty$ for all transreal a except $a = -\infty, \Phi$. We have already confirmed the exceptions in [A4] and [A11], using [A2] where necessary. It remains only to confirm the unexceptional cases.

Using the special addition method we have:

$$a + \infty = \frac{1}{0} + \frac{1}{0} = \frac{1+1}{0} = \frac{2}{0} = \frac{1}{0} = \infty$$

Using the general addition method we have:

$$a + \infty = \frac{n_a}{d_a} + \frac{1}{0} = \frac{n_a \times 0 + d_a \times 1}{d_a \times 0} = \frac{0 + d_a}{0} = \frac{d_a}{0} = \frac{1}{0} = \infty$$

Thus, the axiom is implemented by the methods.

Non-null subtraction of infinity

The *Non-null Subtraction of Infinity* axiom [A10] says $a - \infty = -\infty$ for all transreal a except $a = \infty, \Phi$. The proof is similar to the proof just given, but we set it out for completeness.

We have already confirmed the exceptions in [A4] and [A11], using [A2] and [A9] where necessary. It remains only to confirm the unexceptional cases.

Using the special addition method we have:

$$a - \infty = \frac{-1}{0} - \frac{1}{0} = \frac{-1}{0} + \frac{-1}{0} = \frac{-1 + (-1)}{0} = \frac{-2}{0} = \frac{-1}{0} = -\infty$$

Using the general addition method we have:

$$a - \infty = \frac{n_a}{d_a} - \frac{1}{0} = \frac{n_a}{d_a} + \frac{-1}{0} = \frac{n_a \times 0 + d_a \times (-1)}{d_a \times 0} = \frac{0 + (-d_a)}{0} = \frac{-d_a}{0} = \frac{1}{0} = \infty$$

Thus, the axiom is implemented by the methods.

Bijectivity of reciprocal

The *Bijectivity of Reciprocal* axiom [A5] says $(a^{-1})^{-1} = a$ for all transreal a except $a = -\infty$. We have already confirmed the exception via [A21] and [A20]. We have already confirmed the case $a = \Phi$ via [A22]. We now confirm the remaining cases.

When $-\infty < a \leq \infty$ we have:

$$(a^{-1})^{-1} = (1 \div a)^{-1} = 1 \div (1 \div a) = \frac{1}{1} \div \left(\frac{1}{1} \div \frac{n}{d}\right) = \frac{1}{1} \div \left(\frac{1}{1} \times \frac{d}{n}\right) = \frac{1}{1} \div \left(\frac{1 \times d}{1 \times n}\right)$$
$$= \frac{1}{1} \div \frac{d}{n} = \frac{1}{1} \times \frac{n}{d} = \frac{1 \times n}{1 \times d} = \frac{n}{d} = a$$

Note that for $a > -\infty$ the sign of the denominator propagates to the numerator without loss of information when computing reciprocals.

Positive

The *Positive* axiom [A23] says $\infty \times a = \infty \Leftrightarrow a > 0$. There are four cases to consider: $a < 0, a = 0, a > 0, a = \Phi$. However, the axiom is already confirmed for $a = \Phi$ via [A15] and [A13]. It is also confirmed for $a = 0$ by [A16]. This leaves two cases. Taking $n > 0$ these are as follows.

When $a < 0$ we have:

$$\infty \times a = \frac{1}{0} \times \frac{-n}{d} = \frac{1 \times (-n)}{0 \times d} = \frac{-n}{0} = -\infty$$

When $a > 0$ we have:

$$\infty \times a = \frac{1}{0} \times \frac{n}{d} = \frac{1 \times n}{0 \times d} = \frac{n}{0} = \infty$$

This confirms the axiom.

Negative

The *Negative* axiom [A24] says $\infty \times a = -\infty \Leftrightarrow a > 0$. This is already confirmed by the proof just given, using [A27] where necessary.

Quadrachotomy

The *Quadrachotomy* axiom [A30] says that exactly one of these four relations holds for all transreal a : $a < 0$, $a = 0$, $a > 0$, $a = \Phi$. Part of this, the trichotomy axiom, holds for all real a: $a < 0$, $a = 0$, $a > 0$. Part of this, $a = \Phi$ holds, trivially, for $a = \Phi$. Quadrachotomy holds for ∞ via [A25] and [A23]. It remains only to show that quadrachotomy holds for $-\infty$. Applying [A24] we see:

$$\infty \times -\infty = \frac{1}{0} \times \frac{-1}{0} = \frac{1 \times (-1)}{0} = \frac{-1}{0} = -\infty \Leftrightarrow 0 > -\infty$$

Hence, quadrachotomy applies to $-\infty$. This completes the proof.

Additive associativity

The *Additive Associativity* axiom [A1] says $a + (b + c) = (a + b) + c$ for all transreal a, b, c. There are four, high level, cases to consider. Firstly, when the general addition rule applies to all of the sums. Secondly, when the special addition rule applies to all of the sums. It may also be the case that exactly two of a, b, c may be an infinity so that there are a further three cases: $a, b \in \{-\infty, \infty\}$, $a, c \in \{-\infty, \infty\}$, $b, c \in \{-\infty, \infty\}$. However, the cases with $a, b \in \{-\infty, \infty\}$ and $b, c \in \{-\infty, \infty\}$ are symmetrical so only one of them need be tested. Thirdly, we choose to test $a, b \in \{-\infty, \infty\}$. Finally, we test $a, c \in \{-\infty, \infty\}$. However, if any of $a, b, c = \Phi$ then additive associativity holds via [A4] so that it only remains to test cases involving real numbers and signed infinities. The third and fourth, high level, cases both split into four cases so that there are a total of ten cases to test.

Now that the reader has had some practice using the pencil and paper methods, we apply several of the methods in a single reduction of an equation. We are similarly unfussy in future work.

First, using the general addition method everywhere:

$$
\begin{aligned}
a + (b + c) &= \frac{n_a}{d_a} + \left(\frac{n_b}{d_b} + \frac{n_c}{d_c}\right) = \frac{n_a}{d_a} + \frac{n_b \times d_c + d_b \times n_c}{d_b \times d_c} \\
&= \frac{n_a \times d_b \times d_c + d_a \times (n_b \times d_c + d_b \times n_c)}{d_a \times d_b \times d_c} \\
&= \frac{n_a \times d_b \times d_c + d_a \times n_b \times d_c + d_a \times d_b \times n_c}{d_a \times d_b \times d_c}
\end{aligned}
$$

and

$$
\begin{aligned}
(a + b) + c &= \left(\frac{n_a}{d_a} + \frac{n_b}{d_b}\right) + \frac{n_c}{d_c} = \frac{n_a \times d_b + d_a \times n_b}{d_a \times d_b} + \frac{n_c}{d_c} \\
&= \frac{d_c \times (n_a \times d_b + d_a \times n_b) + n_c \times (d_a \times d_b)}{d_a \times d_b \times d_c} \\
&= \frac{n_a \times d_b \times d_c + d_a \times n_b \times d_c + d_a \times d_b \times n_c}{d_a \times d_b \times d_c}
\end{aligned}
$$

Hence $a + (b + c) = (a + b) + c$ in the first case.

Second, using the special addition method everywhere:

$$a + (b + c) = \frac{n_a}{0} + \left(\frac{n_b}{0} + \frac{n_c}{0}\right) = \frac{n_a}{0} + \frac{n_b + n_c}{0} = \frac{n_a + n_b + n_c}{0} = \frac{n_a + n_b}{0} + \frac{n_c}{0}$$
$$= (a + b) + c$$

Third, $c \notin \{-\infty, \infty, \Phi\}$, that is, $c \in R$.

When $a = \infty$ and $b = \infty$ we have:

$$a + (b + c) = \frac{1}{0} + \left(\frac{1}{0} + \frac{n_c}{d_c}\right) = \frac{1}{0} + \frac{1 \times d_c + 0 \times n_c}{0 \times d_c} = \frac{1}{0} + \frac{d_c}{0} = \frac{1}{0} + \frac{1}{0} = \frac{1 + 1}{0}$$
$$= \frac{2}{0} = \infty$$

and

$$(a + b) + c = \left(\frac{1}{0} + \frac{1}{0}\right) + \frac{n_c}{d_c} = \frac{1 + 1}{0} + \frac{n_c}{d_c} = \frac{2}{0} + \frac{n_c}{d_c} = \frac{1}{0} + \frac{n_c}{d_c} = \frac{1 \times d_c + 0 \times n_c}{0 \times d_c}$$
$$= \frac{d_c}{0} = \frac{1}{0} = \infty$$

So additive associativity holds in this case.

When $a = \infty$ and $b = -\infty$ we have:

$$a + (b + c) = \frac{1}{0} + \left(\frac{-1}{0} + \frac{n_c}{d_c}\right) = \frac{1}{0} + \frac{-1 \times d_c + 0 \times n_c}{0 \times d_c} = \frac{1}{0} + \frac{-d_c}{0} = \frac{1}{0} + \frac{-1}{0}$$
$$= \frac{1 + (-1)}{0} = \frac{0}{0} = \Phi$$

and

$$(a + b) + c = \left(\frac{1}{0} + \frac{-1}{0}\right) + \frac{n_c}{d_c} = \frac{1 + (-1)}{0} + \frac{n_c}{d_c} = \frac{0}{0} + \frac{n_c}{d_c} = \Phi$$

So additive associativity holds in this case.

When $a = -\infty$ and $b = \infty$ we have:

$$a + (b + c) = \frac{-1}{0} + \left(\frac{1}{0} + \frac{n_c}{d_c}\right) = \frac{-1}{0} + \frac{1 \times d_c + 0 \times n_c}{0 \times d_c} = \frac{-1}{0} + \frac{d_c}{0} = \frac{-1}{0} + \frac{1}{0} = \frac{-1+1}{0}$$
$$= \frac{0}{0} = \Phi$$

and

$$(a + b) + c = \left(\frac{-1}{0} + \frac{1}{0}\right) + \frac{n_c}{d_c} = \frac{-1+1}{0} + \frac{n_c}{d_c} = \frac{0}{0} + \frac{n_c}{d_c} = \Phi$$

So additive associativity holds in this case.

When $a = -\infty$ and $b = -\infty$ we have:

$$a + (b + c) = \frac{-1}{0} + \left(\frac{-1}{0} + \frac{n_c}{d_c}\right) = \frac{-1}{0} + \frac{-1 \times d_c + 0 \times n_c}{0 \times d_c} = \frac{-1}{0} + \frac{-d_c}{0} = \frac{-1}{0} + \frac{-1}{0}$$
$$= \frac{(-1) + (-1)}{0} = \frac{-2}{0} = -\infty$$

and

$$(a + b) + c = \left(\frac{-1}{0} + \frac{-1}{0}\right) + \frac{n_c}{d_c} = \frac{(-1) + (-1)}{0} + \frac{n_c}{d_c} = \frac{-2}{0} + \frac{n_c}{d_c} = \frac{-1}{0} + \frac{n_c}{d_c}$$
$$= \frac{-1 \times d_c + 0 \times n_c}{0 \times d_c} = \frac{-d_c}{0} = \frac{-1}{0} = -\infty$$

So additive associativity holds in this case. This completes the third, high level, case.

Fourth, $b \notin \{-\infty, \infty, \Phi\}$, that is, $b \in R$. This case is similar to the third, high level, case, just proved. But, as these proofs have not been given before, there is some value in setting them out explicitly.

When $a = \infty$ and $c = \infty$ we have:

$$a + (b + c) = \frac{1}{0} + \left(\frac{n_b}{d_c} + \frac{1}{0}\right) = \frac{1}{0} + \frac{n_b \times 0 + d_c \times 1}{d_c \times 0} = \frac{1}{0} + \frac{d_c}{0} = \frac{1}{0} + \frac{1}{0}$$
$$= \frac{1 + 1}{0} = \frac{2}{0} = \frac{1}{0} = \infty$$

and

$$(a + b) + c = \left(\frac{1}{0} + \frac{n_b}{d_c}\right) + \frac{1}{0} = \frac{1 \times d_c + 0 \times n_b}{0} + \frac{1}{0} = \frac{d_c}{0} + \frac{1}{0} = \frac{1}{0} + \frac{1}{0}$$
$$= \frac{1 + 1}{0} = \frac{2}{0} = \frac{1}{0} = \infty$$

So additive associativity holds in this case.

When $a = \infty$ and $c = -\infty$ we have:

$$a + (b + c) = \frac{1}{0} + \left(\frac{n_b}{d_b} + \frac{-1}{0}\right) = \frac{1}{0} + \frac{n_b \times 0 + d_b \times (-1)}{d_b \times 0} = \frac{1}{0} + \frac{-d_b}{0} = \frac{1}{0} + \frac{-1}{0}$$
$$= \frac{1 + (-1)}{0} = \frac{0}{0} = \Phi$$

and

$$(a + b) + c = \left(\frac{1}{0} + \frac{n_b}{d_b}\right) + \frac{-1}{0} = \frac{1 \times d_b + n_b \times 0}{0 \times d_b} + \frac{-1}{0} = \frac{d_b}{0} + \frac{-1}{0} = \frac{1}{0} + \frac{-1}{0}$$
$$= \frac{1 + (-1)}{0} = \frac{0}{0} = \Phi$$

So additive associativity holds in this case.

When $a = -\infty$ and $c = \infty$ we have:

$$a + (b + c) = \frac{-1}{0} + \left(\frac{n_b}{d_b} + \frac{1}{0}\right) = \frac{-1}{0} + \frac{n_b \times 0 + d_b \times 1}{d_b \times 0} = \frac{-1}{0} + \frac{d_b}{0} = \frac{-1}{0} + \frac{1}{0}$$
$$= \frac{-1 + 1}{0} = \frac{0}{0} = \Phi$$

and

$$(a+b)+c = \left(\frac{-1}{0}+\frac{n_b}{d_b}\right)+\frac{1}{0} = \frac{-1\times d_b+0\times n_b}{0\times d_b}+\frac{1}{0} = \frac{-d_b}{0}+\frac{1}{0} = \frac{-1}{0}+\frac{1}{0}$$
$$= \frac{-1+1}{0} = \frac{0}{0} = \Phi$$

So additive associativity holds in this case.

When $a = -\infty$ and $c = -\infty$ we have:

$$a+(b+c) = \frac{-1}{0}+\left(\frac{n_b}{d_c}+\frac{-1}{0}\right) = \frac{-1}{0}+\frac{n_b\times 0+d_c\times(-1)}{d_c\times 0} = \frac{-1}{0}+\frac{-d_c}{0} = \frac{-1}{0}+\frac{-1}{0}$$
$$= \frac{(-1)+(-1)}{0} = \frac{-2}{0} = \frac{-1}{0} = -\infty$$

and

$$(a+b)+c = \left(\frac{-1}{0}+\frac{n_b}{d_c}\right)+\frac{-1}{0} = \frac{-1\times d_c+0\times n_b}{0}+\frac{-1}{0} = \frac{-d_c}{0}+\frac{-1}{0} = \frac{-1}{0}+\frac{-1}{0}$$
$$= \frac{(-1)+(-1)}{0} = \frac{-2}{0} = \frac{-1}{0} = -\infty$$

So additive associativity holds in this case. This completes the fourth, high level, case and completes the entire proof.

Distributivity

The *Distributivity* axiom [A31] says, when a, b, c are transreal it is the case that $a\times(b+c) = (a\times b)+(a\times c) : \neg((a = \pm\infty)\wedge(\text{sgn}(b)\neq\text{sgn}(c))\wedge(b+c\neq 0, \Phi))$.

Distributivity holds when a, b, c are real. If any of $a, b, c = \Phi$ then distributivity holds via [A4]. It remains to test the cases where at least one of a, b, c is infinite and the remaining terms, if any, are real.

First, suppose that a is real.

When the special method of addition applies we have:

$$a\times(b+c) = \frac{n_a}{d_a}\times\left(\frac{n_b}{0}+\frac{n_c}{0}\right) = \frac{n_a}{d_a}\times\frac{n_b+n_c}{0}$$

This distributes when $\frac{n_b + n_c}{0} = \frac{0}{0}$. Otherwise we continue as follows:

$$\frac{n_a}{d_a} \times \frac{n_b + n_c}{0} = \frac{n_a \times (n_b + n_c)}{d_a \times 0} = \frac{n_a \times n_b + n_a \times n_c}{0}$$

and

$$(a \times b) + (a \times c) = \left(\frac{n_a}{d_a} \times \frac{n_b}{0}\right) + \left(\frac{n_a}{d_a} \times \frac{n_c}{0}\right) = \frac{n_a \times n_b}{d_a \times 0} + \frac{n_a \times n_c}{d_a \times 0} = \frac{n_a \times n_b}{0} + \frac{n_a \times n_c}{0}$$

This distributes when $\frac{n_a \times n_b}{0} = \frac{0}{0}$ or $\frac{n_a \times n_c}{0} = \frac{0}{0}$. Otherwise we continue as follows:

$$\frac{n_a \times n_b}{0} + \frac{n_a \times n_c}{0} = \frac{n_a \times n_b + n_a \times n_c}{0}$$

So distributivity holds in this case.

When the general method of addition applies, exactly one of $b = \pm\infty$ or $c = \pm\infty$, but as the equations are symmetrical in b, c we need test only one of them:

$$a \times (b + c) = \frac{n_a}{d_a} \times \left(\frac{n_b}{0} + \frac{n_c}{d_c}\right) = \frac{n_a}{d_a} \times \frac{n_b \times d_c + 0 \times n_c}{0 \times d_c} = \frac{n_a}{d_a} \times \frac{n_b \times d_c}{0} = \frac{n_a \times n_b \times d_c}{d_a \times 0}$$
$$= \frac{n_a \times n_b}{0}$$

and

$$(a \times b) + (a \times c) = \left(\frac{n_a}{d_a} \times \frac{n_b}{0}\right) + \left(\frac{n_a}{d_a} \times \frac{n_c}{d_c}\right) = \frac{n_a \times n_b}{d_a \times 0} + \frac{n_a \times n_c}{d_a \times d_c} = \frac{n_a \times n_b}{0} + \frac{n_a \times n_c}{d_a \times d_c}$$
$$= \frac{n_a \times n_b \times d_a \times d_c + 0 \times n_a \times n_c}{0 \times d_a \times d_c} = \frac{n_a \times n_b \times d_a \times d_c}{0} = \frac{n_a \times n_b}{0}$$

So distributivity holds in this case. This completes the first, high level, case where a is real.

The remaining cases all have $a = \pm\infty$.

When $b + c = 0$ we have, without loss of generality, $c = -b$ and $b, c \in R$:

$$a \times (b + c) = \pm\infty \times 0 = \frac{\pm 1}{0} \times \frac{0}{1} = \frac{\pm 1 \times 0}{0 \times 1} = \frac{0}{0}$$

and

$$(a \times b) + (a \times c) = \left(\frac{\pm 1}{0} \times \frac{n_b}{d_b}\right) + \left(\frac{\pm 1}{0} \times \frac{-n_b}{d_b}\right) = \frac{\pm 1 \times n_b}{0 \times d_b} + \frac{\pm 1 \times (-n_b)}{0 \times d_b} = \frac{\pm n_b}{0} + \frac{\mp n_b}{0}$$

This distributes when $\frac{\pm n_b}{0} = \frac{0}{0}$. Otherwise we continue as follows:

$$\frac{\pm n_b}{0} + \frac{\mp n_b}{0} = \frac{\pm n_b + \mp n_b}{0} = \frac{0}{0}$$

So distributivity holds in this case. This confirms the part of the guarding clause which says $b + c \neq 0$.

When $b + c = \Phi$ we have, without loss of generality, $b = -c$ so that $c = \mp\infty$ when $b = \pm\infty$. In the case we are examining, $a = \pm\infty$ regardless of the signs of b, c. Now:

$$a \times (b + c) = \frac{\pm 1}{0} \times \left(\frac{\pm 1}{0} + \frac{\mp 1}{0}\right) = \frac{\pm 1}{0} + \frac{(\pm 1) + (\mp 1)}{0} = \frac{\pm 1}{0} + \frac{0}{0} = \frac{\pm 1 \times 0 + 0 \times 0}{0 \times 0} = \frac{0}{0}$$

and

$$(a \times b) + (a \times c) = \left(\frac{\pm 1}{0} \times \frac{\pm 1}{0}\right) + \left(\frac{\pm 1}{0} \times \frac{\mp 1}{0}\right) = \frac{\pm 1}{0} + \frac{\mp 1}{0} = \frac{(\pm 1) + (\mp 1)}{0} = \frac{0}{0}$$

So distributivity holds in this case. This confirms the part of the guarding clause which says $b + c \neq \Phi$.

When $\text{sgn}(b) = \text{sgn}(c)$ we have, without loss of generality, $b = \frac{\pm n_b}{d_b}$ and $c = \frac{\pm n_c}{d_c}$ such that $n_b, n_c > 0$. In the case we are examining, $a = \pm\infty$ regardless of the signs of b, c. Now:

$$\begin{aligned} a \times (b + c) &= \frac{\pm 1}{0} \times \left(\frac{\pm n_b}{d_b} + \frac{\pm n_c}{d_c}\right) = \frac{\pm 1}{0} \times \frac{\pm n_b \times d_c + d_b \times \pm n_c}{d_b \times d_b} \\ &= \frac{\pm 1 \times (\pm n_b \times d_c + d_b \times \pm n_c)}{0 \times d_b \times d_b} = \frac{\pm 1 \times (\pm n_b \times d_c + d_b \times \pm n_c)}{0} \\ &= \frac{\pm 1 \times (\pm n_b + \pm n_c)}{0} = \frac{(\pm 1 \times \pm n_b) + (\pm 1 \times \pm n_c)}{0} \end{aligned}$$

and

$$\begin{aligned} (a \times b) + (a \times c) &= \left(\frac{\pm 1}{0} \times \frac{\pm n_b}{d_b}\right) + \left(\frac{\pm 1}{0} \times \frac{\pm n_c}{d_c}\right) = \frac{\pm 1 \times \pm n_b}{0 \times d_b} + \frac{\pm 1 \times \pm n_c}{0 \times d_c} \\ &= \frac{\pm 1 \times \pm n_b}{0} + \frac{\pm 1 \times \pm n_c}{0} = \frac{(\pm 1 \times \pm n_b) + (\pm 1 \times \pm n_c)}{0} \end{aligned}$$

So distributivity holds in this case. This confirms the part of the guarding clause which says $\text{sgn}(b) \neq \text{sgn}(c)$.

It remains to show that the guarding clause does guard against non-distributivity in every case where $(a = \pm\infty) \wedge (\text{sgn}(b) \neq \text{sgn}(c)) \wedge (b + c \neq 0, \Phi)$. Without loss of generality, we may choose $b > c$. There are then three cases to consider.

When b is positive and c is zero:

$$\begin{aligned} a \times (b + c) &= \frac{\pm 1}{0} \times \left(\frac{n_b}{d_b} + \frac{0}{1}\right) = \frac{\pm 1}{0} \times \frac{n_b \times 1 + d_b \times 0}{d_b \times 1} = \frac{\pm 1}{0} \times \frac{n_b}{d_b} = \frac{\pm 1 \times n_b}{0 \times d_b} = \frac{\pm 1 \times n_b}{0} \\ &= \frac{\pm 1}{0} \end{aligned}$$

and

$$(a\times b)+(a\times c) = \left(\frac{\pm 1}{0}\times\frac{n_b}{d_b}\right)+\left(\frac{\pm 1}{0}\times\frac{0}{1}\right) = \frac{\pm 1\times n_b}{0\times d_b}+\frac{\pm 1\times 0}{0\times 1} = \frac{\pm 1\times n_b}{0}+\frac{0}{0}$$
$$= \frac{\pm 1}{0}+\frac{0}{0} = \frac{\pm 1\times 0+0\times 0}{0\times 0} = \frac{0}{0}$$

So distributivity does not hold in this case, as required.

When b is positive and c is negative we may choose $d = b+c>0$:

$$a\times(b+c) = a\times d = \frac{\pm 1}{0}\times\frac{n_d}{d_d} = \frac{\pm 1\times n_d}{0\times d_d} = \frac{\pm 1\times n_d}{0} = \frac{\pm 1}{0}$$

and

$$(a\times b)+(a\times c) = \left(\frac{\pm 1}{0}\times\frac{n_b}{d_b}\right)+\left(\frac{\pm 1}{0}\times\frac{-n_c}{d_c}\right) = \frac{\pm 1\times n_b}{0\times d_b}+\frac{\pm 1\times -n_c}{0\times d_c}$$
$$= \frac{\pm 1\times n_b}{0}+\frac{\pm 1\times -n_c}{0} = \frac{\pm 1}{0}+\frac{\mp 1}{0} = \frac{(\pm 1)+(\mp 1)}{0} = \frac{0}{0}$$

So distributivity does not hold in this case, as required.

When b is zero and c is negative:

$$a\times(b+c) = \frac{\pm 1}{0}\times\left(\frac{0}{1}+\frac{-n_c}{d_c}\right) = \frac{\pm 1}{0}\times\frac{0\times d_c+1\times -n_c}{1\times d_c} = \frac{\pm 1}{0}\times\frac{-n_c}{d_c} = \frac{\pm 1\times -n_c}{0\times d_c}$$
$$= \frac{\pm 1\times -n_c}{0} = \frac{\mp 1}{0}$$

and

$$(a\times b)+(a\times c) = \left(\frac{\pm 1}{0}\times\frac{0}{1}\right)+\left(\frac{\pm 1}{0}\times\frac{-n_c}{d_c}\right) = \frac{\pm 1\times 0}{0\times 1}+\frac{\pm 1\times -n_c}{0\times d_c} = \frac{0}{0}+\frac{\pm 1\times -n_c}{0}$$
$$= \frac{0}{0}+\frac{\mp 1}{0} = \frac{0\times 0+0\times\mp 1}{0\times 0} = \frac{0}{0}$$

So distributivity does not hold in this case, as required. This completes the proof.

Proof that the pencil and paper methods implement nothing but the axioms of transreal arithmetic

We have just shown that the pencil and paper methods do implement the axioms of transreal arithmetic, but now we want to know that everything that the methods calculate, can be obtained from the axioms. Proving this from first principles would be much more work than we undertook in the previous section, but we can take a short cut. We could perform a census of how all of the arithmetical operations of *multiplication, division, addition,* and *subtraction* apply to any transreal arguments; and of how the predicates *positive* and *negative* apply to any transreal arguments. But the methods tell us that *division* is *multiplication by the reciprocal* and that *subtraction* is *addition of the opposite* so we need not perform a census of *division* and *subtraction.* Further, we can derive *positive* and *negative* from the relation *greater than.* This means we can conduct the census in just three tables for: *multiplication, addition,* and *greater than.* See.[5] In the tables q is a positive, rational number; n is a positive numerator; and d is a non-negative denominator. The letter T stands for *T*rue, F for *F*alse, and C means *C*onditionally true or false, identically with the case for real arithmetic.

The work of constructing the tables is simply the work of performing the requisite arithmetic using the methods. To be fair, this is still more work than the we did in the previous section, but at least the calculations are very simple. The actual work is left as an exercise for the reader.

$n_1/d_1 \times n_2/d_2$		$-q_2$	0	q_2	$-\infty$	∞	Φ
		$(-n_2)/d_2$	$0/1$	n_2/d_2	$(-1)/0$	$1/0$	$0/0$
$-q_1$	$(-n_1)/d_1$	q_3	0	$-q_3$	∞	$-\infty$	Φ
0	$0/1$	0	0	0	Φ	Φ	Φ
q_1	n_1/d_1	$-q_3$	0	q_3	$-\infty$	∞	Φ
$-\infty$	$(-1)/0$	∞	Φ	$-\infty$	∞	$-\infty$	Φ
∞	$1/0$	$-\infty$	Φ	∞	$-\infty$	∞	Φ
Φ	$0/0$	Φ	Φ	Φ	Φ	Φ	Φ

Table 3.1: Multiplication of transreal numbers

$n_1/d_1 + n_2/d_2$		$-q_2$	0	q_2	$-\infty$	∞	Φ
		$(-n_2)/d_2$	$0/1$	n_2/d_2	$(-1)/0$	$1/0$	$0/0$
$-q_1$	$(-n_1)/d_1$	$-q_3$	$-q_1$	$\pm q_3$	$-\infty$	∞	Φ
0	$0/1$	$-q_2$	0	q_2	$-\infty$	∞	Φ
q_1	n_1/d_1	$\pm q_3$	q_1	q_3	$-\infty$	∞	Φ
$-\infty$	$(-1)/0$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	Φ	Φ
∞	$1/0$	∞	∞	∞	Φ	∞	Φ
Φ	$0/0$	Φ	Φ	Φ	Φ	Φ	Φ

Table 3.2: Addition of transreal numbers

$n_1/d_1 > n_2/d_2$		$-q_2$	0	q_2	$-\infty$	∞	Φ
		$(-n_2)/d_2$	$0/1$	n_2/d_2	$(-1)/0$	$1/0$	$0/0$
$-q_1$	$(-n_1)/d_1$	C	F	F	T	F	F
0	$0/1$	T	F	F	T	F	F
q_1	n_1/d_1	T	T	C	T	F	F
$-\infty$	$(-1)/0$	F	F	F	F	F	F
∞	$1/0$	T	T	T	T	F	F
Φ	$0/0$	F	F	F	F	F	F

Table 3.3: Greater than

Having constructed the tables by using the methods, we now need to show that every entry of the table can be derived from the axioms. Again, this is a very great deal of work, but it has already been done for us.[6] The reader who wishes to see how theorems are derived from the axioms of transreal arithmetic is invited to read.[6]

Verifying alternative methods

Now that we know that the above tables follow from the axioms, this gives us a simpler way of verifying that any methods we care to develop do implement all and only the axioms of transreal arithmetic. First, show that division is multiplication by the reciprocal and that subtraction is addition of the opposite. Second, verify that the new methods give the same results as shown in the tables. Third, verify that the new methods do not give any results that are not in the tables.

Conclusion

The chapter starts by giving, in a few pages, pencil and paper methods for performing transreal arithmetic. This section also indicates how ordinary arithmetic can be taught which outlaws division by zero, how ordinary arithmetic can be taught so that it does not obstruct the learning of division by zero, and how division by zero can be taught. This gives teachers the flexibility to outlaw transreal arithmetic or to encourage it in school teaching. If the majority of teachers use their existing skills, but stop using methods which obstruct division by zero, and specialist mathematics teachers teach division by zero, then the transfer from teaching ordinary arithmetic to teaching transreal arithmetic can be made smoothly. But this transition should only be undertaken in a school if and when its teachers are comfortable with transreal arithmetic. But, as transreal arithmetic contains ordinary arithmetic, and is consistent with it, it could be taught to just the most able pupils as a method of deepening their understanding of ordinary mathematics. Again, only a confident and committed teacher should undertake the teaching of transreal arithmetic in this way.

The rest of the chapter proves that the pencil and paper methods work. But the methods are not necessarily the most efficient, nor the easiest to teach to children, nor the methods with the least difference from what is currently taught in schools. It will take some engagement with teachers and pupils to be sure that the methods given are good ones. All I can say is that they are the best I have developed so far. But note that I am dyslexic. My needs and coping strategies are not necessarily indicative of how the majority of people do arithmetic.

The section which shows how the methods implement the axioms of transreal arithmetic is rather long. I could make the derivations shorter by using more compact notation and by taking bigger steps, but, in my view, this would be a hindrance to the reader who is not already fluent in transreal arithmetic. I believe the constant repetition of derivations is a useful pedagogic device, all be it a rather dull one. On the other hand, I should probably make the section longer by leading the reader more gently through the rather complicated branching structure of some of the proofs.

I welcome feedback from the reader on all of the above points. And I especially welcome feedback that corrects any errors I have made. Now, let me share a little personal history with you.

The youngest person to learn transreal arithmetic

A woman came to work at the University last week. She met me on her second day and we transacted our business. At the end of the meeting she asked how I divide by zero so, in the next half hour, I showed her the methods that I have now written up in this chapter. She went home and taught the methods to her eleven year old son that evening. He went to school the next day and started to teach the methods to his mathematics teacher, but ran out of time in a single lesson. The teacher could not believe the methods, but could find no fault in them. To date, this eleven year old is the youngest person to learn transreal arithmetic.

CHAPTER 4

The Graph of the Reciprocal

Introduction

A great many people have attempted to criticise transreal arithmetic by exploiting the asymmetry that the reciprocal of both positive and negative infinity is zero, but the reciprocal of zero is positive infinity and not negative infinity. This is not, as many critics suppose, a bug in the specification of transreal arithmetic, it is a feature. The asymmetry plays an important role in retaining the maximum possible amount of information about the sign and magnitude of numbers. All of the approaches, promulgated by my critics, preserve no more information than transreal arithmetic does, and many of them lose information.

Bottom

Bottom, $\perp$, has several uses in mathematics. It is used as the first element of any type of thing. For example, the bottom element of the non-negative integers is zero, the bottom element of the positive integers is one, the bottom element of men is Adam (citing biblical sources) and the bottom element of bananas is the very first banana that ever grew. Thus, bottom is a non-negative integer, a positive integer, a man, a banana, and everything else that has a type.

A second use of bottom is as an element in logic which means that no information is known about a thing.[14] For example, if a calculation on the non-negative integers produces $x = \perp$ as the result, it means that no information is known about the number x. It might be that x is zero, one, two, three, or any non-negative integer at all, including being just bottom. But if one failed to specify that the domain of interest is non-negative integers, then x might be a man, a banana, or any type of thing. When bottom is used in arithmetic, it is quite common to give it ordering properties that do not disturb the ordinary ordering of numbers, and to define that it is of type *number* and of type *error*. This allows bottom to be used as a number everywhere in arithmetic and to be treated as an error in computer programs.

Very commonly, authors do not bother to say what type(s) bottom has so that one must read into their work a sensible interpretation. In what follows, we refrain from saying what type(s) bottom has and leave the reader to interpret our use of bottom.

Bottom can be defined in various ways in arithmetic. One popular way is to define that $\perp = n/0$ for all numbers n. This leads to the graph of the reciprocal sketched below.

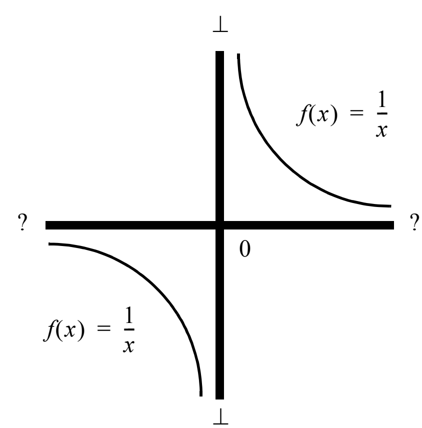

Figure 4.1: Bottom as the reciprocal of zero

The graph is drawn in two pieces, one in the first quadrant of the Cartesian co-ordinate frame (top-right) and the other in the third quadrant (bottom-left). Both pieces are governed by the equation $f(x) = 1/x$. At $x = 0$ we have $f(x) = f(0) = 1/0 = \perp$. We are free to choose whether bottom is a point high up on the y-axis, the vertical axis, or low down, or both, or neither. We simply have no information about its position. If we want to say that the graph of $f(x)$ is continuous at zero, so that it is made out of a single piece, joined at a single point, bottom, then we are free to do so. But we are equally free to say that the graph falls into two, or more, pieces. We could say that the graph of $f(x)$ falls into four pieces: the arc in the first quadrant, the arc in the third quadrant, bottom high up in the graph, and bottom low down in the graph. We have a very great deal of freedom to say where bottom lies.

If we use bottom, we are still free to use an unsigned infinity, ∞, two signed infinities, $-\infty, +\infty$, or any number of infinities or other things. Depending on the choices we make, we may say that the graph of $f(x)$ asymptotes to zero as it moves to the right. That is to say, it gets closer and closer to zero, but never arrives at zero. Or we may say that it does arrive at zero. And we may specify that $f(x)$ has the same, or different, behaviour at the left and right of the graph. We could even specify that the graph arrives at bottom at the extreme left and right. This is a perfectly symmetrical solution, but the symmetry is bought by throwing away information about the sign and magnitude of $f(x)$ at $x = 0$ and at the extreme right-hand and extreme left-hand values of x, whatever they may be. Quite simply, bottom gives no information. But we can retain some information about the magnitude of a number by using an unsigned infinity.

Unsigned infinity

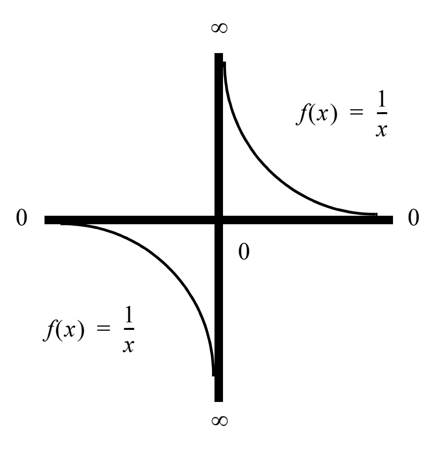

Figure 4.2: Unsigned infinity as the reciprocal of zero

A popular way to use an unsigned infinity is to define $\infty = n/0$ for all non-zero numbers, n. We are then free to define $1/\infty = 0$. The figure above shows a sketch of the graph of $f(x) = 1/x$ as one piece. The arc in the first quadrant is connected to the arc in the second quadrant at the unsigned infinity, ∞. The graph at $x = 0$ looks like it is in two places, but we define that unsigned infinity is one place.

As we move to the extreme right, $x = \infty$ and $f(x) = f(\infty) = 1/\infty = 0$ so that the arc makes contact with the x-axis of the co-ordinate frame. Similarly, at the extreme left, $x = -\infty$ and $f(x) = f(-\infty) = f(\infty) = 1/\infty = 0$ so that the arc also makes contact with the x-axis of the co-ordinate frame here. Note that $-\infty = \infty$ because infinity is unsigned. In the same way, $-0 = 0$ because zero is unsigned.

With this arrangement we know that $f(0) = \infty$ has a very large magnitude, but we do not know anything about its sign. We do not know if infinity is positive, negative, or whether it has any sign at all. This infinity is useful in some areas of mathematics, but it does not give us as much information about the reciprocal as we would get by using a signed infinity.

Signed infinity

In what follows, we write positive infinity without a sign, ∞, and negative infinity with a sign, $-\infty$.

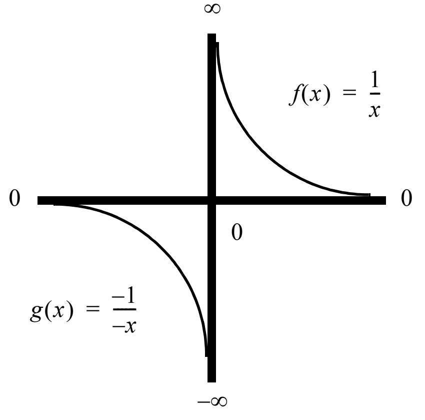

Figure 4.3: Signed infinity as the reciprocal of zero

The graph of the reciprocal is in two pieces. The piece in the first quadrant is governed by the equation $f(x) = 1/x$. The piece in the third quadrant is governed by $g(x) = -1/(-x)$. Using transreal arithmetic we have:

$$f(0) = \frac{1}{0} = \infty \qquad \text{[E 4.1]}$$

$$f(\infty) = \frac{1}{\infty} = \frac{1}{1} \div \frac{1}{0} = \frac{1}{1} \times \frac{0}{1} = \frac{0}{1} = 0 \qquad \text{[E 4.2]}$$

$$g(0) = \frac{-1}{-0} = \frac{-1}{0} = -\infty \qquad \text{[E 4.3]}$$

$$g(-\infty) = \frac{-1}{-(-\infty)} = \frac{-1}{\infty} = \frac{-1}{1} \div \frac{1}{0} = \frac{-1}{1} \times \frac{0}{1} = \frac{0}{1} = 0 \qquad \text{[E 4.4]}$$

Thus, we have complete knowledge of the reciprocal of x for all real numbers x, extended by the signed infinities $-\infty, \infty$. But we can obtain even more information.

Let us define the reciprocal, r, as follows.

$$r(x) = \begin{cases} \frac{1}{x} & : x \not< 0 \\ \frac{-1}{-x} & : x \not> 0 \end{cases} \qquad \text{[D1]}$$

Here $x \not< 0$ means that x is not less than zero. This may be because $x \geq 0$ or $x = \Phi$. Similarly $x \not> 0$ means that x is not greater than zero. This may be because $x \leq 0$ or $x = \Phi$.

We have already examined the cases $r(x) = f(x)$ when $x \geq 0$, and $r(x) = g(x)$ when $x \leq 0$. It only remains to examine the cases when $x = \Phi$.

$$r(\Phi) = f(\Phi) = \frac{1}{\Phi} = \frac{1}{1} \div \frac{0}{0} = \frac{1}{1} \times \frac{0}{0} = \frac{0}{0} = \Phi \qquad \text{[E 4.5]}$$

$$r(\Phi) = g(\Phi) = \frac{-1}{-\Phi} = \frac{-1}{\Phi} = \frac{-1}{1} \div \frac{0}{0} = \frac{-1}{1} \times \frac{0}{0} = \frac{0}{0} = \Phi \qquad \text{[E 4.6]}$$

In transreal arithmetic the sign of nullity is nullity and the magnitude of nullity is nullity so we now have complete knowledge of the sign and magnitude of the reciprocal of any transreal number.

Conclusion

One must take care to define the reciprocal so that it applies to all of the transreal numbers. See definition [D1]. When this is done the reciprocal of every transreal number can be found. Which is to say that the transreciprocal is a total function that preserves full knowledge of the sign and magnitude of numbers. It is simply not sufficient to take the real-number definition of the reciprocal and to adopt this without change. Real numbers do not deal with infinite and non-finite objects so we cannot reasonably assume that real-number definitions will continue to hold for all transreal numbers. We must be prepared to extend real definitions to transreal definitions before we use them to solve mathematical problems.

Exercises

4.1 There are many reciprocals that use bottom. Choose one of these and define a function that maps the transreciprocal onto the reciprocal of your choice. In other words, show how to cripple the transreciprocal so that it is a reciprocal with bottom.

See Answer 4.1

4.2 There are many reciprocals that use an unsigned infinity. Choose one of these and define a function that maps the transreciprocal onto the reciprocal of your choice. In other words, show how to cripple the transreciprocal so that it is a reciprocal with an unsigned infinity.

See Answer 4.2

4.3 The following argument[2] is sometimes presented as a fallacious counter-proof of the possibility of dividing by zero. Say what is wrong with it.

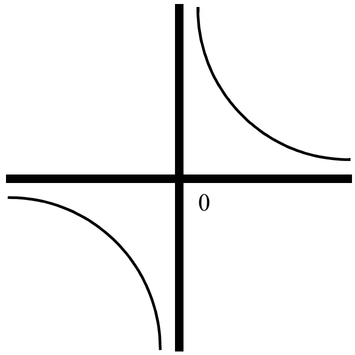

Figure 4.4: Fallacious reciprocal

At first glance it seems possible to define $a/0$ by considering the limit a/b as b approaches zero.

For any positive a it is known that:

$$\lim_{b \to 0^+} \frac{a}{b} = +\infty$$

For any negative a it is known that:

$$\lim_{b \to 0^+} \frac{a}{b} = -\infty$$

Therefore $a/0$ is defined to be $+\infty$ when a is positive and as $-\infty$ when a is negative.

However, taking the limit from the right is arbitrary. The limits could be taken from the left. In which case $a/0$ is defined to be $-\infty$ when a is positive and as $+\infty$ when a is negative.

This can be further illustrated using the following equation (when it is assumed that several properties of the real numbers apply to the infinities)

$$+\infty = \frac{1}{0} = \frac{1}{-0} = -\frac{1}{0} = -\infty$$

Which leads to $+\infty = -\infty$ which would be a contradiction with the standard definition of the extended real-number line. The only workable extension is to introduce an unsigned infinity.

Furthermore there is no obvious definition of $0/0$ that can be derived by considering the limit of a ratio.

The limit:

$$\lim_{(a,\, b) \to (0,\, 0)} \frac{a}{b}$$

does not exist.

Limits of the form

$$\lim_{x \to 0} \frac{f(x)}{g(x)}$$

in which both of $f(x)$ and $g(x)$ approach zero, as x approaches zero, may equal any real or infinite value, or may not exist at all, depending on the particular functions f and g. These and other, similar, facts show that $0/0$ cannot be well defined as a limit.

See Answer 4.3

CHAPTER 5

NaN

Introduction

Most of the world's general purpose computers use the IEEE standard for floating-point arithmetic.[12] This standard allows a computer to divide any number by zero so it is legitimate to ask how transreal arithmetic differs from IEEE floating-point arithmetic. Let us begin by examining the views of those who believe that the two approaches are very similar, before examining the standard for ourselves, and coming to a conclusion about which approach is more satisfactory.

Wikipedia

As I write this chapter, in April 2009, Wikipedia has a biographical entry on me.[3] This entry argues that my proposals are very close to those of the IEEE standard. This view is entirely mistaken, but it is a useful exercise to examine it in detail.

I have deleted hypertext links from this, and subsequent, quotations from Wikipedia, and have corrected spellings that are incorrect in both British and American English, but I have not modified the punctuation of the quotations. I have changed the style of mathematical symbols so that it is consistent with the rest of this book.

> *Anderson quickly gained publicity in December 2006 in the United Kingdom when the regional BBC South Today reported his claim of "having solved a 1200 year old problem", namely that of division by zero. However,*

> *commentators quickly pointed out that his ideas are just a variation of the standard IEEE 754 concept of NaN (Not a Number), a datum that has been commonly employed on computers in floating point arithmetic for many years. Dr. Anderson defended the criticism of his claims on BBC Berkshire on 12 December 2006, saying, "If anyone doubts me I can hit them over the head with a computer that does it."*

The quotation begins:

> *Anderson quickly gained publicity in December 2006 in the United Kingdom when the regional BBC South Today reported his claim of "having solved a 1200 year old problem", namely that of division by zero.*

This is substantially true. I did gain a great deal of publicity, very quickly, when the BBC's regional, television, news program *South Today* reported my visit to a local school and this was picked up by the BBC's global, television, news programme *BBC 24.* There were 90 000 hits on my web site that day, with another 10 000 hits the next day. In the following fortnight 40 000 people downloaded a copy of the paper giving the axioms of transreal arithmetic,[6] a figure which continues to grow by about 500 downloads per month. In a typical month the site gets 3 000 hits, but this jumps to 10 000 hits per month when I publish something new. The quotation correctly reports my words, but it is arguable, on historical grounds, whether people have sought to divide by zero for 1200 or 1300 years. Either way, the historical fault is mine. I was being cautious in the radio programme.

> *However, commentators quickly pointed out that his ideas are just a variation of the standard IEEE 754 concept of NaN (Not a Number), a datum that has been commonly employed on computers in floating point arithmetic for many years.*

It is true that commentators, including some of my friends, did say this. But I argue, in this chapter, that transreal arithmetic is a radical departure from the IEEE standard, both in its mathematical content and in its consequences for theoretical and practical computing. It should be noted, however, that the quotation is wrong on one point. *NaN* is not a datum. It is a set of data, a point to which I return, below.

> *Dr. Anderson defended the criticism of his claims on BBC Berkshire on 12 December 2006, saying, "If anyone doubts me I can hit them over the head with a computer that does it."*

It is true that I defended my claims, and I continue to do so. The quotation is, however, reported out of context. It was put to me that it is impossible to represent infinity with a binary code. This is false. For example, the IEEE standard does it.[12] As I already had a computer that performed transreal arithmetic,[7] it seemed quite reasonable, to me, to face my critics with a brute fact – here is a computer that does arithmetic on infinity with binary arithmetic. But no listener took me up my offer to see the computer (which was subsequently dismantled because its components were needed for a laboratory class). There are, however, several living witnesses who saw the computer in operation.

The Wikipedia article[3] also says:

> *He has written two papers on division by zero and has invented what he calls the "Perspex machine".*

The number of papers continues to grow and I have, indeed, invented a computer that I call the Perspex machine. In future, I hope it will be on sale to the public, and that all general purpose computers will use transreal arithmetic.

> *Anderson claims that "mathematical arithmetic is sociologically invalid" and that IEEE floating-point arithmetic, with NaN, is also faulty.*

I do, indeed, make these claims, and defend them in this chapter.

> *Anderson's transreal numbers were first mentioned in a 1997 publication, and made well-known on the Internet in 2006, but not accepted as useful by the mathematics community. These numbers are used in his concept of transreal arithmetic and the Perspex machine. According to Anderson, transreal numbers include all of the real numbers, plus three others: infinity (∞), negative infinity ($-\infty$) and "nullity" (Φ), a numerical representation of a non-number that lies outside of the affinely extended real number line. (Nullity, confusingly, has an existing mathematical meaning.)*

The reference to the 1997 publication[4] is correct, in so far as it refers to the first mention of the point at nullity. Nullity is dealt with in a geometrical construction, and is produced as the solution to some equations, but it is not arithmetised until a later publication. As there are about 100 000 web pages on my work, dated 2006, and only my pages before that, it is correct to say that my work was made well know on the world wide web in 2006. The mathematics community certainly has not accepted the usefulness of the transreal numbers. I do not expect pure mathematicians to be concerned with usefulness; and it is entirely normal, and logically necessary, for applied mathematicians to lag behind the scientific fields, such as Computer Science, Physics and Chemistry, where their talents are applied. However, no professional mathematician, so far as I am aware, now challenges the correctness of transreal arithmetic. It is true that I say that positive infinity, negative infinity, and nullity are numbers. But I have never made the self-contradictory claim that the number nullity represents a non-number. Though I do claim that the number nullity lies off the extended, real-number line. It is true that nullity has other mathematical meanings, but so do many mathematical terms. For example, “zero” may be an integer, a rational number, a real number, a complex number, a quaternion, an octonion, an element of a ring, an element of a group, or an element of various other algebraic structures. Each of these different kinds of zero has some properties in common, but may have some properties that are not common. For example, some of these zeros are less than one, and some are not. This is not confusing to a mathematician, it is a consequence of the wide variety of, and interrelation between, mathematical concepts. But, if anyone can think of a better name than *nullity* or a better symbol than Φ then I will listen to their views.

> *Anderson intends the axioms of transreal arithmetic to complement the axioms of standard arithmetic; they are supposed to produce the same result as standard arithmetic for all calculations where standard arithmetic defines a result. In addition, they are intended to define a consistent numeric result for the calculations which are undefined in standard arithmetic, such as division by zero.*

Correct.

> *In standard arithmetic, division of zero by zero is undefined. In calculus, it is an indeterminate form.*

Correct.

> *Transreal arithmetic resembles IEEE floating point arithmetic, a floating point arithmetic commonly used on computers. In IEEE floating point arithmetic, calculations such as zero divided by zero can produce a result, Not a Number (NaN), to which the standard arithmetic axioms do not apply (as it is not a number). IEEE floating point arithmetic, like transreal arithmetic, uses affine infinity (two separate infinities, one positive and one negative) rather than projective infinity (a single unsigned infinity, turning the number line into a loop). The IEEE standard extends standard arithmetic by defining the results of all arithmetic operations upon* $-\infty$*,* $+\infty$*, and NaN.*

I deny that transreal arithmetic resembles IEEE floating-point arithmetic, and set out my denial in this chapter. The rest of the quotation is uncontentious, except for the last sentence:

> *The IEEE standard extends standard arithmetic by defining the results of all arithmetic operations upon* $-\infty$*,* $+\infty$*, and NaN.*

If "all arithmetic operations" means the binary operations of addition, subtraction, multiplication, division, finding the remainder, ordering relations, and the unary operation of computing the square root of a non-negative object or minus zero, then the quotation is true. See,[12] pages 7 and 11. The standard does not define an extension of unary negation, but in commentary on the standard, it notes that $-x \neq 0 - x$ (see,[12] page 17). By contrast, it is a theorem of transreal arithmetic, which is lexically identical to a theorem of ordinary arithmetic, that $-x = 0 - x$. The IEEE standard extends the operation of taking the square root, extracting a remainder, rounding to integer, optionally rounding to some other precisions (see,[12] page 10), and specifies various copying and testing predicates (see,[12] especially page 11). But no other mathematical functions are extended. The standard is silent, for example, on how to find cube roots, how to find logarithms and exponents, how to compute trigonometric functions, or how to compute any mathematical function, other than those explicitly listed in the present paragraph, and in the 10 pages of the standard,[12] pages 7 - 16, excluding front and back matter.

> *Loosely put, IEEE floating point arithmetic extends the (affinely) extended real number line to the set* $R \cup \{-\infty, \mathrm{NaN}, \infty\}$ *and transreal arithmetic extends the*

> *(affinely) extended real number line to the set* $R \cup \{-\infty, \Phi, \infty\}$. *IEEE floating point arithmetic approximates this set, defining a finite binary representation, for actual use in computers, that includes denormal numbers and negative zero. Transreal arithmetic has no standardized binary encoding for use in computers.*

Why say, "loosely put?" This is an indication that what is about to be said is false. The quotation goes on to say that floating-point arithmetic does not deal with the set of real numbers, R, but only with a finite subset of these numbers. I will allow this as a legitimate looseness in expression, but the claim that IEEE floating-point arithmetic is defined on the set $R \cup \{-\infty, \mathrm{NaN}, \infty\}$ is contradicted by the correct claim that the standard defines an object -0. See,[12] pages 12 and 13. The IEEE object $-0 = 0$, but -0 and 0 are different objects, for example, $1 \div 0 = \infty$ but $1 \div (-0) = -\infty$. In contrast, transreal arithmetic has $-0 = 0$ with -0 and 0 being the same object so that $1 \div (-0) = 1 \div 0 = \infty$. Making this correction, by including the object negative zero, we now have that IEEE floating-point arithmetic is defined on the set $R \cup \{-\infty, -0, \mathrm{NaN}, \infty\}$, but this contains a category error. *NaN* is not an object, it is a set of objects. In 32-bit, single precision, floating-point arithmetic there are $2^{23} - 1 = 8\,388\,607$ *NaN* objects; and in single precision, extended arithmetic there are a minimum of $2^{32} - 1 = 4\,294\,967\,295$ *NaN* objects; in 64-bit, double precision, arithmetic there are $2^{52} - 1 = 4\,503\,599\,627\,370\,495$ *NaN* objects; and in double precision, extended, arithmetic there are at least $2^{64} - 1 = 18\,446\,744\,073\,709\,551\,615$ *NaN* objects. The standard recommends,[12] page 10, that all implementations of the standard should support the extended format at the highest precision of basic format that is supported. Making this correction, we now have that IEEE floating-point arithmetic is defined on the set $R \cup \{-\infty, -0, \mathrm{NaN}_i, \infty\}$ with i being at least $i = 4\,294\,967\,295$ on 32-bit machines, and, on 64-bit machines, being at least $i = 18\,446\,744\,073\,709\,551\,615$. Counting negative zero in with *NaN* this is exactly 4G differences in 32-bit arithmetic and 16E differences in 64-bit arithmetic. The standard defines that no *NaN* object is equal to any floating-point number, nor is it equal to any *NaN* object, including itself (see,[12] page 12), so all of the *NaN* objects are distinct objects. By contrast, transreal arithmetic[6] has $\Phi = 0/0$ is a unique number, and $\Phi = \Phi$. Speaking loosely, transreal arithmetic has eighteen quintillion differences from IEEE floating-point arithmetic.

As an aside, if anyone seriously maintains that *NaN* and nullity are the same thing, or even resemble each other, then I invite them to lend me £4,294,967,295 and accept repayment, in full, with £1. I am willing to do this any number of times, with any number of people. But I am far more generous than this. I am willing to nett £18,446,744,073,709,551,614 on a single trade, except that there is not this much money in the entire world, even when all the currencies in the world are converted to the pound sterling at their current rate. Has the penny dropped yet? *NaN* and nullity are very different things.

It is true that there is no standard, binary representation of the transreal numbers; but there are non-standard representations. For example, see.[7] Perhaps, one day, transreal numbers will be standardised by the computing community.

> *Just as in IEEE floating point arithmetic, in transreal arithmetic, all calculations including infinity and nullity are axiomatically defined. In addition, several of the axioms of standard arithmetic are constrained so that they do not operate upon the extended members of the number set. Here are some identities in transreal arithmetic with the IEEE equivalents:*

Correct. But the table[3] is almost completely wrong. I have added the table legend.

Transreal arithmetic	**IEEE floating-point arithmetic**
$+1 \div 0 = \infty$	$+1 \div +0 = \infty$
$-1 \div 0 = -\infty$	$+1 \div -0 = -\infty$
$0 \div 0 = \Phi$	$0 \div 0 = \text{NaN}$
$\infty \times 0 = \Phi$	$\infty \times 0 = \text{NaN}$
$\infty - \infty = \Phi$	$\infty - \infty = \text{NaN}$
$\Phi + a = \Phi$	$\text{NaN} + a = \text{NaN}$
$\Phi \times a = \Phi$	$\text{NaN} \times a = \text{NaN}$
$-\Phi = \Phi$	$-\text{NaN} = \text{NaN}$ i.e. applying unary negation to NaN yields NaN
$\Phi = \Phi \Rightarrow \text{TRUE}$	$\text{NaN} = \text{NaN} \Rightarrow \text{FALSE}$

Table 5.1:Fallacious comparison of Transreal versus IEEE floating-point arithmetic

The transreal column is correct with the usual reading of the mathematical symbols, but the IEEE floating-point column is incoherent. It uses at least three different, and incompatible, readings of the equality symbol, "=." Following the Principle of Charity we try to make the best argument we can for the Wikipedia author. The first correction is to correct the category error by replacing the un-indexed *NaN* with an appropriately indexed *NaN*. The second correction is to distinguish two, incompatible, readings of the equality sign. We do this by producing two, corrected, IEEE columns: one recording equality as $a = b$ and the other recording the computation of a result (a return value) as $a \rightarrow b$. In the equality column we use separate indexes (*i, j*) within a row, because no *NaN* is equal to any *NaN,* including itself (see,[12] page 12). In the assignment column we use separate indexes (*i, j*) because the kind of *NaN* that is returned depends, amongst other things, on whether or not an appropriate trap is set (see,[12] pages 13 and 14). The third correction is to remove a third, incompatible, reading of the equality sign by replacing $\Phi = \Phi \Rightarrow \text{TRUE}$ with $\Phi = \Phi$ and by replacing $\text{NaN} = \text{NaN} \Rightarrow \text{FALSE}$ by $\text{NaN} \neq \text{NaN}$. In the IEEE standard it is a mutually exclusive option whether relational operations return a Boolean value, TRUE or FALSE, or whether they return no value at all and set flags. See,[12] page 12. Again, following the Principle of Charity, we make the least disruptive assumptions we can and allow the Boolean relations. With these corrections in place, we do not need to annotate the IEEE column with the sentence, "i.e. applying unary negation to *NaN* yields *NaN,"* because our correct use of the equals sign does not require any extraordinary explanation.

Transreal arithmetic	**IEEE Floating-point arithmetic**	
Equality and Return Value	**Equality**	**Return Value**
$+1 \div 0 = \infty$	$+1 \div +0 = \infty$	$+1 \div +0 \rightarrow \infty$
$-1 \div 0 = -\infty$	$+1 \div -0 = -\infty$	$+1 \div -0 \rightarrow -\infty$
$0 \div 0 = \Phi$	$0 \div 0 \neq \text{NaN}_i$	$0 \div 0 \rightarrow \text{NaN}_i$
$\infty \times 0 = \Phi$	$\infty \times 0 \neq \text{NaN}_i$	$\infty \times 0 \rightarrow \text{NaN}_i$
$\infty - \infty = \Phi$	$\infty - \infty \neq \text{NaN}_i$	$\infty - \infty \rightarrow \text{NaN}_i$
$\Phi + a = \Phi$	$\text{NaN}_i + a \neq \text{NaN}_j$	$\text{NaN}_i + a \rightarrow \text{NaN}_j$
$\Phi \times a = \Phi$	$\text{NaN}_i \times a \neq \text{NaN}_j$	$\text{NaN}_i \times a \rightarrow \text{NaN}_j$

Transreal arithmetic	IEEE Floating-point arithmetic	
Equality and Return Value	**Equality**	**Return Value**
$-\Phi = \Phi$	$-\text{NaN}_i \neq \text{NaN}_j$	$-\text{NaN}_i \rightarrow \text{NaN}_j$
$\Phi = \Phi$	$\text{NaN}_i \neq \text{NaN}_j$	$\text{NaN}_i \rightarrow \text{NaN}_j$

Table 5.2:Corrected comparison of Transreal versus IEEE floating-point arithmetic

On the most charitable reading, only the first row of the IEEE *Equality* column shows the same behaviour as transreal arithmetic, that is, $+1 \div +0 = \infty$. The remaining eight rows show a different behaviour. Again, on the most charitable reading of the IEEE *Return Value* column, if we ignore the category error, by ignoring the indexes on *NaN,* then eight of the rows are the same, and only the second row is different: $-1 \div 0 = -\infty$ but $+1 \div -0 \rightarrow -\infty$. On a less generous reading, if we pay attention to the indexes on *NaN,* then only the first row is the same, $+1 \div +0 \rightarrow \infty$, and the remaining eight rows are different. In summary, if we ignore the category error then 9 out of 18 comparisons are different, but if we pay attention to the indexes on *NaN* then 16 out of 18 comparisons are different.

> *The main difference between transreal arithmetic and IEEE floating-point arithmetic is that whilst nullity compares equal to nullity, NaN does not compare equal to NaN.*

Correct.

> *Due to the more expansive definition of numbers in transreal arithmetic, several identities and theorems which apply to all numbers in standard arithmetic are not universal in transreal arithmetic. For instance, in transreal arithmetic, $a - a = 0$ is not true for all a, since $\Phi - \Phi = \Phi$. That problem is addressed in [one of Anderson's papers]. Similarly, it is not always the case in transreal arithmetic that a number can be cancelled with its reciprocal to yield 1. Cancelling zero with its reciprocal in fact yields nullity.*

Correct.

Examining the axioms provided by Anderson, it is easy to see that any term which contains an occurrence of the constant Φ is provably equivalent to Φ. Formally, let t be any term with a sub-term Φ, then $t = \Phi$ is a theorem of the theory proposed by Anderson.

This is a correct description of transreal arithmetic, but this behaviour of nullity does not generalise to all transnumber systems. Nullity is not a universal bottom element which automatically drives all terms containing it to bottom, as a future publication will demonstrate.

I hope I have now done enough to demonstrate that the Wikipedia article[3] on the similarity of transreal arithmetic[6] and IEEE floating-point arithmetic[12] has no merit. As I have already pointed out more than eighteen quintillion differences, I am at a loss to know how else to criticise the article, in its own terms, so I now turn to criticising the central thesis of the article by examining the IEEE standard itself. Again, we shall see that there is no merit in the suggestion that transreal arithmetic and IEEE floating-point arithmetic are similar. I begin by examining the ordering relations.

IEEE standard

Predicate	**Greater**	**Less**	**Equal**	**Unordered**	**Exception**
=	F	F	T	F	No
?<>	T	T	F	T	No
>	T	F	F	F	Yes
>=	T	F	T	F	Yes
<	F	T	F	F	Yes
<=	F	T	T	F	Yes
?	F	F	F	T	No
<>	T	T	F	F	Yes
<=>	T	T	T	F	Yes
?>	T	F	F	T	No
?>=	T	F	T	T	No
?<	F	T	F	T	No
?<=	F	T	T	T	No
?=	F	F	T	T	No

Predicate	Greater	Less	Equal	Unordered	Exception
NOT(>)	F	T	T	T	Yes
NOT(>=)	F	T	F	T	Yes
NOT(<)	T	F	T	T	Yes
NOT(<=)	T	F	F	T	Yes
NOT(?)	T	T	T	F	No
NOT(<>)	F	F	T	T	Yes
NOT(<=>)	F	F	F	T	Yes
NOT(?>)	F	T	T	F	No
NOT(?>=)	F	T	F	F	No
NOT(?<)	T	F	T	F	No
NOT(?<=)	T	F	F	F	No
NOT(?=)	T	T	F	F	No

Table 5.3:IEEE ordering relations: 14 positive, 12 negations, 12 exceptions

The IEEE standard,[12] pages 12 and 13, provides four, mutually exclusive, Boolean, ordering relations: *less than* (<), *equal* (=), *greater than* (>), and *unordered* (?). As special cases, *minus zero* and *zero* compare equal (-0 = 0), even though these two objects are different, and *NaN* objects compare unequal, even if they are identical $\mathrm{NaN}_i \neq \mathrm{NaN}_i$. Apart from these special cases, the relations *less than, equal,* and *greater than* all have their usual mathematical meanings. The *unordered* relation is true (*T*) if any of its arguments is *NaN,* and is false (*F*) otherwise. This gives the only standard way of determining if an object, x, is *NaN:* by testing the truth of $x?x$. The forms *isnan*(*x*) and $x \neq x$ are specifically excluded from the standard. See,[12] page 17. While the four ordering relations are mutually exclusive, they are not orthogonal: there are 14 positive relations, with no *NOT* predicate, and 12 negations, with *NOT* predicates. The non-negated pair of relations is equal (=) and not equal (?<>). Consequently, the missing negations are *NOT*(=) and *NOT*(?<>). Precisely 12 of the Boolean relations generate exceptions (error conditions) if any of their arguments is *NaN.* But the implementor of the standard is free to choose whether to supply Boolean operations, with exceptions, or else flags – *greater, less, equal, unordered* – without exceptions. If Boolean relations are implemented then only the first 6 are mandatory (=, ?<>, >, >=, <, <=). If exceptions are generated, the programmer can choose whether to handle the exceptions in a trap, or else to let the standard complying system follow its default behaviour. All in all, the IEEE standard specifies two, complicated, means of ordering floating-point numbers and allows considerable variation in how these are implemented.

Transreal arithmetic[6] provides three, mutually exclusive, Boolean, ordering relations: *less than* (<), *equal* (=), and *greater than* (>). These relations have their usual mathematical meaning. There are no special cases and no exceptions. The operations are orthogonal, with no missing predicates and no missing negations. The empty symbol () with no occurrences of <, =, > is not listed because it is empty. The full symbol (<=>) with all occurrences of <, =, > is not used listed because it is constant true. Nonetheless, these symbols could be supported by a computer language, if desired. The implementor is free to implement transreal ordering relations with Boolean predicates, flags, or any sufficient method.

Predicate	**Greater**	**Less**	**Equal**
=	F	F	T
>	T	F	F
>=	T	F	T
<	F	T	F
<=	F	T	T
<>	T	T	F
NOT(=)	T	T	F
NOT(>)	F	T	T
NOT(>=)	F	T	F
NOT(<)	T	F	T
NOT(<=)	T	F	F
NOT(<>)	F	F	T

Table 5.4:Transreal ordering relations: 6 positive, 6 negations

In summary, transreal arithmetic provides the usual mathematical ordering relations, with their ordinary mathematical meanings. But IEEE floating-point arithmetic provides an additional (and completely redundant) mathematical ordering relation (*unordered*), changes the meaning of the ordinary mathematical relation of equality, changes the other relational operations so that some of them generate exceptions, omits two negations, forbids the implementor from using both flags and Boolean relations, and, perversely, recommends that *NaN*s are identified by non-standard methods. All of these departures from the ordinary mathematical ordering relations are redundant, as the counter example of transreal arithmetic demonstrates.

The IEEE standard,[12] pages 13 and 14, specifies two kinds of *NaN*s – signalling *NaN*s and quiet *NaN*s. Either kind of *NaN* may be used for any purpose, except that signalling *NaN*s must be reserved values and must raise an exception – which may or may not be handled by the programmer. If the programmer chooses not to handle the exception, and a floating-point result is to be returned, then a quiet *NaN* must be returned (thereby discharging the signal). The standard does not say which quiet *NaN* should be returned. If an operation is applied to some quiet *NaN*s, but no signalling *NaN*s, and a floating-point result is to be returned, then one of the argument *NaN*s must be returned. But the standard is silent on which one should be returned. In either case, the return value must be in a basic or extended format that is capable of representing the required *NaN*. Hence, the existence of *NaN*s places an obligation on the language designer to ensure that the return value is large enough. By contrast, transreal arithmetic[6] has no silent or signalling objects of any kind. It has only unique numbers, leaving the language designer free to implement any sufficient type system.

The IEEE standard,[12] page 13, specifies infinity in terms of real limits, but this specification cannot be achieved, as this counter-example shows. Suppose that $P_i = P$ is a representably large, positive, floating-point number such that $2P$ is unrepresentably large and rounds off to infinity in the default rounding mode. Then $\lim_{i \to \infty} P_0 + P_1 - P_2 - P_3 + k_4 + 0_5 + \ldots + 0_i = k$ for any real k, but the floating-point sum rounds off to infinity. Now, real $k \neq \infty$ and we have a contradiction. By contrast, transreal arithmetic defines infinity as the unique number $\infty = 1/0$, irrespective of the existence of limits. But, to be fair to the IEEE standard, it also gives another specification of infinity that can be achieved. Presumably, one is supposed to ignore the impossible specification.

The IEEE standard,[12] pages 14 and 15, specifies five exceptions (error conditions). These are: *invalid operation, division by zero, overflow, underflow, inexact.* But in transreal arithmetic it is not an error to divide by zero so the *division by zero* exception is pointless. Similarly, transreal arithmetic is a total arithmetic, meaning that any defined, arithmetical operation can be applied to any operands. Hence, the *invalid operation* exception can never occur. These exceptions are redundant.

In summary, the IEEE standard[12] defines an *unordered* relation which is redundant, and which greatly confuses the ordering relations by introducing special cases and exceptions (error states). It also introduces two redundant exceptions: *invalid operation* and *division by zero*. Transreal arithmetic has none of these redundancies, special cases, or exceptions.

I hope I have now done enough to show that there is no merit in the thesis that transreal arithmetic resembles IEEE floating-point arithmetic. There is a great deal more I could say on the standard, but it pales into insignificance when we compare the roles of transreal arithmetic and the IEEE standard.

Scientific status

Transreal arithmetic is a scientific theory that, under the evolutionary pressure of maintaining efficiency in a physical mind described by perspexes, certain historical algorithms of arithmetic evolved to allow division by zero in a way which is mathematically consistent[6] and which is consistent with the behaviour of the physical universe.[7] Quite separately from this theory, the axioms of transreal arithmetic have been proved to be consistent,[6] but the proof is, potentially, falsifiable: either through some error in the proof or by a revision in the meaning of the terms used in the proof. Independently of challenges to the axiomatic system, the underlying theory can be challenged by developing other transnumber systems and by using all of the transnumber systems to make scientific predictions in physics, engineering, and the like. Once a transnumber system has been axiomatised, mathematicians can make predictions from the mathematical theory. By contrast, IEEE floating-point arithmetic is not a scientific theory, it is a standard agreed by a committee. The standard was known to be inconsistent shortly after it was introduced. The inconsistencies were resolved by adopting certain popular computers as reference systems, and by comparing all future floating-point systems to these. It is possible to make mathematical predictions about the performance of floating-point algorithms, given a common understanding of the norm-referenced behaviour of floating-point arithmetic, but the standard can be modified at any time, and can lapse or be withdrawn at its quinquennial reviews. The IEEE standard is a commercial document, not a scientific theory. It is no surprise, therefore, that when the standard is judged by scientific criteria it fails. Occam's razor militates in favour of transreal arithmetic because it uses the ordinary mathematical ordering relations, including equality, with no special cases or error states. By contrast, IEEE floating-point arithmetic uses a redundant ordering relation (unordered), introduces special cases of equality, and introduces numerous redundant objects (*NaN*s). IEEE floating-point arithmetic is grossly inefficient in its ordering of numbers, a fault which has commercial implications.

Mathematical status

Historically, mathematics has progressed by formalising certain concepts used in domestic life, commerce, industry, and science. The formalisations are then proved, extended, and developed into abstract mathematical theories. It is easy to forget this history when surveying the panoply of modern mathematics, but history is impressed on the present. The multiplicative inverse is a formalisation of several ways of performing division. It is built deep into the foundations of algebra, calculus, and topology. But it is only one way of formalising division. Division with remainder is another formalisation of division which illuminates different, and therefor compatible, aspects of division. But transreal division raises a more serious challenge. It is consistent with the multiplicative inverse, wherever this inverse is defined, but it also applies in cases where the multiplicative inverse does not apply. It places extra demands on mathematical proof and algorithms, but it repays this extra cost by making mathematics total. Some mathematicians have examined total systems, but the majority of practical mathematics is partial – it fails on division by zero and its mathematical consequences. It is a fact of life, attested to by the widespread use of IEEE floating-point arithmetic, that ordinary mathematics does not meet the needs of domestic, commercial, industrial, and scientific computing. In this sociological sense, ordinary mathematics is invalid. It does not describe what people do when they compute. Transreal arithmetic is an attempt to improve on all pre-existing social and arithmetical approaches to dividing by zero.

Commercial status

The IEEE standard is embodied in general-purpose computer chips with an annual market of, about, 150 Bn US dollars. Computer systems built using these chips have a considerably higher commercial value. The social impact of these systems is huge. The IEEE floating-point standard, produced by a US standards body, is very complicated and is norm-referenced to computers manufactured in the US. It provides a barrier to the entry of new chip suppliers into a market which is, currently, dominated by one supplier – the US company, Intel. Proposed revisions of the IEEE standard increase its complexity and raise the hurdle on entry to the general-purpose chip market. The US can be rightly proud of its commercial and technological innovation. Its chip manufacturers now have an opportunity to adopt transreal arithmetic which will make both its floating-point and integer processors[7] more efficient. But its competitors have the same opportunity. An opportunity which is magnified by the software market for reliable and safe software based on a simpler computer arithmetic.

I wonder how this will turn out. Will the US continue to develop silicon guzzling chips at a time when other countries develop lean technologies? And will the outcome of this scenario be any different from the development of gas guzzling automobiles?

CHAPTER 6

Sheila Roberts

Introduction

Sheila Roberts (1920-) was a teacher trainer, specialising in mathematics, throughout much of her career. She was involved in the introduction of the new mathematics in the 1960s. I was interested in her experiences of introducing a new mathematical syllabus because it might have important lessons for the introduction of transarithmetic, or some other total arithmetic, into primary and secondary schools. I met Sheila on 26 February 2008 and made written notes and an audio recording of the meeting. The text below is an edited version of those records which has been corrected by Sheila. It conveys those parts of our discussion that seem relevant to a future introduction of a mathematics syllabus. Sheila begins by recalling her move from being a teacher in a girl's grammar school to being a teacher trainer in a college in the 1960s. She had also been a teacher in a boy's school and a mixed-sex school.

Sheila – structure of teacher training courses

In 1960 all teacher training courses were increased from two years to three years, and on passing the course students gained a Certificate of Education not a degree. The degree qualification came later. Students who trained for primary teaching would have a three year course in educational studies and professional studies, mathematics being a compulsory element, and an academic study, at their own level, in a subject of their own choice. So one of my jobs was to bring the students into some sort of comprehension of what a child can cope with in mathematics. In the first term of their second year they would have a period of time with me, and each student would go into a school equipped with their own material. They made everything in those days. They made their rulers …

James They made their own rulers?

Sheila - home made mathematical equipment Yes, I'll show you one.

[Sheila produced a yard stick made out of a light coloured wood. On one side alternate foot lengths were marked in red, on the other side alternate inches were marked in red. The stick was obviously home made.]

James Gosh.

Sheila They could get any help they liked. They could go to a brother or a carpenter or a father. And that was the yard rule I used. Of course that did them good, because they had to think about it.

James I have seen things like this, but made more professionally. What about things like set squares and protractors?

Sheila Protractors no, because angle wasn't taught in junior school. Except that they would make a surveying instrument with a straw and a plumb line. And that was better than a professional instrument because the children could make it themselves. There was a lot of "hands on" with children.

Sheila – understanding before rote learning Anyway, each student would take two children for about an hour on a topic we would agree with the school that they would do. The school might say, "Well, we will leave it to you," so I would let the student choose. Depending on the age of the children they might do weighing or number or measurement. It was very hands on. Our policy was that children should understand, before you start pushing rote learning. They don't figure that way now.

James What do they figure now?

Sheila They concentrate on rote learning.

What did give teaching a bad name was people said, "Oh, they don't have to learn tables." But they do have to learn tables. But our policy was to learn how a table is built up before they had to do rote learning.

James	What was the motivation for teaching tables? One possibility is that it is so that people can do arithmetic. Another is that it is the basis of multiplication tables in algebra. Was that a consideration at primary school or was it purely practical?
Sheila – mathematics as a practical skill	That's a good question … I remember now. It was practical. They had to use what they learnt. We had an awful lot of practical work. Students would take children shopping. I don't think there was much pressure on, "they will need it later in algebra." We didn't call it "arithmetic." We called it "mathematics." We also introduced geometry.
James	What was the motive for introducing geometry. I am surprised you had to *introduce* it.
Sheila – geometry	It was looking at shape. Looking at the world around us. We worked with other departments. In science they might meet shapes that are strong, like the triangle.
James	There are a lot of triangles in the environment when you come to think of it.
Sheila	I used to go to college by train. And I used to come into Clapham Junction and they had cranes. And I remember sitting in the train window when it stopped and thinking, this is what students need to see, this is what a child needs to see. The boom of a crane is made out of girders arranged in triangles. We did encourage taking children to see things in reality.
James	A school would do visits?
Sheila	Yes. That's when they had the money.
James	I went to a village primary school where I was the ninth pupil. We went on trips out, but it was all rural. There was nothing industrial to see.
Sheila – rural schools	I musn't digress. But my brother left industrial dairying and became a teacher. He ran a school farm for junior-middle in the seventies, and he had in his school: cows, sheep, ducks. And the children did everything. They let them out in the morning. They weighed their food. They recorded the number of eggs. At weekends my brother was gradually able to ease off and the children would run everything. The eggs were sold in a local market.
James	Where was this?

Sheila	It was in Buckinghamshire.
James	So that was a rural setting, was it?
Sheila	Yes. It was.
	At least the children knew that milk doesn't come from a supermarket, it comes from a cow.
James	Can you tell me more about what was taught in schools?
Sheila – algebra	There was some algebra. Guess the hidden number, that sort of thing.
James	The solution of an equation with a single variable?
Sheila – new mathematics	Yes, yes.
	[Sheila produced a book[8] she had written to help teachers with the new mathematics. She pointed to the cover.]
	That was the sort of thing we had to have. One coloured child, a boy and a girl. You can see the sort of equipment they had.
	I think there was a surge to get children to understand more. We did get some schools that were reluctant. But there was enormous encouragement from the Local Education Authority. Again they had the money. Teachers would come all day Friday, and we would do stuff like in that book. And they would take it back to the school the next week. Then, when they came back again, they had to say how they had introduced some new work and what had happened. And, particularly, where they ran into difficulties.
James	And did that help?
Sheila – fear of mathematics	Well, I thought that it did. We had some teachers who were terrified of the subject. It is terrible to think that we had some teachers who were terrified of mathematics, isn't it?
James	Do you think that has changed?
Sheila	I hope it has.

James	There were some teachers who were *terrified?*
Sheila	Perhaps I am exaggerating. Frightened. Particularly about, "What you are going to do with us," "I only know what I do," if you know what I mean?
James	So teachers would have a limited range of things they could do, which was based on their own experience. They didn't have any depth behind it, or know what the mathematics was for, or why it works?
Sheila – communicating the purpose of mathematics	Exactly. This is personal. But I had a tutor when I did my postgraduate training. I was a mature student, because of the war. This was at secondary school. But he said that if a child said, "Why are we doing this?" then you knew you had done a bad bit of teaching.
James	Where did the new mathematics come from?
Sheila – universities were a repository of new mathematics	Universities. And it must have come from their concern about the incoming students not knowing enough of the modern approach and content.
James	And modern content would mean things like set theory and …
Sheila – Boolean algebra	That's right. And Boolean algebra.
James	Was that seen as a basis for logic or for the computer industry?
Sheila	Logic in those days. Computers were just beginning in the 1960s.
James	Did you tie it up with Venn diagrams and sets?
Sheila – Venn diagrams	Yes, yes.
James	So that is giving quite a pictorial, solid, picture of logic by tying it to the diagrams.
Sheila	Yes. I think it gelled very well with the hands-on approach. The surge was, that we, as teachers, didn't know the modern maths.

James That raises a lot of logistical problems. How were all the teachers taught, because there are many tens of thousands of teachers. Who did the teaching? How was it financed? How long did it take?

Sheila – teacher training for the new mathematics I'll tell you. Kings College London ran courses for us, once a week, on Monday evening from six to eight. This would be for us, the teacher trainers. We were taught by the university staff. Everybody was terribly keen.

The mathematics section of the Teacher Training Association ran a summer conference, for a week, in the summer vacation. The morning was given over to algebra, computing, or whatever it was. The computer was put in for a bit of a lark. In the afternoon we would have another lecture, and sometimes one in the evening. The schools inspectors came too, as participants and lecturers.

We got an enormous amount of support from the universities.

I loved those courses. I started to take off.

The Local Education Authority put advisors into schools to help the teachers with the new mathematics. That was the way they saw the future. That help would come from within the schools. Universities couldn't do everything.

James What I am seeing is, there were a lot of staff in the universities who already knew the new mathematics. They passed it on to the people doing the teacher training, and there were advisors within a school who would help their colleagues.

Sheila – books The other thing, James, was that there were an enormous number of books being published. Which is again because there was the money then.

James Where did all this money come from?

Sheila – school inspectors I think it is when Harold Macmillan said we never had it so good.

I had a very high regard for the school inspectors. I had a particular situation. I was teaching at a grammar school. The head teacher wanted me to teach mathematics to the children who had given up on it. I had them calculate what it would cost to decorate a bed sit, like they might have when they left college. They had to go to the shops to get realistic prices. And that was an enormous success the first year. Then the inspection year, I get a rebellious group. You do in schools. I really thought, "what rotten luck." I couldn't get anything from them. The inspector, she became a good friend in later years, she said, "I think you write them off." She told the Head, and it was down in the report. But, she said, "I musn't leave it at that. I'll give my mind to this and ask my colleagues what to do." And she turned up in six months time and made some very good suggestions which she discussed with me and the Head teacher.

I was on the staff of a college, independent of any Local Education Authority. Our college was next door to a Roman Catholic college, down the road was a Church of England college, and across the road was a Methodist college. The four of us were not Local Authority, we were independent.

James It's interesting. There were religious teacher training colleges. Were they training teachers for their own schools or more generally?

Sheila – faith schools Well, I suppose, once, when they started, for their own schools. But eventually, generally. There were still Catholic, Methodist, and Church of England schools who only took their own. But that was going. There was a loosening up.

James Did all of the religious denominations pursue the new mathematics? Was there any religious dimension of encouragement or restriction?

Sheila Everybody taught the new mathematics. The church authorities governed their colleges, not the content of courses. They would say things like, "Are you doing the new mathematics?" that sort of thing, but nothing more than that. They didn't say they wouldn't and they didn't necessarily give any encouragement.

James We haven't discussed examinations yet.

Sheila – examinations My involvement was the initiation of the Bachelor of Education degree. We set the questions, but they were vetted by the university.

The universities had quite an influence on the examination boards for A-level in secondary schools.

We had some problems. Some students had done the new mathematics and some had not.

James How did you cope with having students with different backgrounds?

Sheila We would put on special courses. And, as far as examination questions were concerned, they would be either or, for a while.

It's a different life now. In my day it was security of tenure. You remained there as long as you wanted, and could cope.

James

So let's explore that for a moment. School teacher had tenure or …

Sheila – tenure

Yes. School teachers had tenure.

James

That's interesting. University teachers used to have tenure. That was removed in about 1982. I had always known that university staff had tenure. The idea was that it gave them the security to challenge existing knowledge. That was the motivation for it. You would know that you were secure in your job if you went around challenging all of the accepted notions. Otherwise you might come under pressure …

Sheila – influence of the British Empire

… and you are out.

In a sense Harold Macmillan was right. We never had it so good.

And the other thing was that we were loosing our Empire. My Principal was written to by the British Council, saying that since we had so many overseas students, they would be very pleased if somebody would like to have secondment to go overseas to see what jobs our students had when they went back home. Like a cheeky Charlie, I decided I'd quite like to do it, even though I had only been at the college a short while, and I went to Ghana. And what Ghana wanted was new mathematics. The Americans were printing books on new mathematics for Africa.

James

How did the Ghanaians know that they wanted new mathematics, rather than just mathematics?

Sheila

I know. When they were part of the Empire they were taking English examinations, so when they got their independence they knew what we were doing.

James

You say the Americans printed books on new mathematics. Was new mathematics being introduced in America at the same time?

Sheila – influence of the space race

Yes, of course. Don't forget Sputnik, which the Russians got into space first. After that, the Americans decided that mathematics must be taught vigorously.

James

So how long did it take, in Britain, before everyone switched over to the new mathematics?

Sheila – speed of introduction of new mathematics

Probably about five years. It was quite quick, but the examinations pushed it through.

James

I am told, but I don't know how accurate this is, that, these days, text books in a school turn over in about fifteen years. So to switch a mathematics syllabus in five, if things were funded at the same rate, must have involved a special effort to change the text books.

Sheila

Yes. There were a heck of a lot of books being published. It really seized everybody. We were influenced by Sputnik.

James

I have heard mathematics teachers say, these days, that people are happy to say they are innumerate.

Sheila – innumeracy

That is very true, actually.

I don't mind saying I am a mathematician now.

They'll say, "I was never good, but my daughter was quite good." "My daughter gets on with it," or "isn't worried by it."

Children who have said, "Mathematics frightened me." They only had to have an off day, in the mathematics class, that they would break the sequence, coming through. They would have to jump over something they hadn't comprehended in the beginning, and were thrust into the next stage, before they had really done the earlier work.

James

Now that is a little bit surprising. Mathematics is largely hierarchical, but it is not nearly as hierarchical as people make out.

Sheila – non-hierarchical nature of mathematics

No, it isn't.

James

But if, by missing a single lesson, you could break the sequence, it means there wasn't much revision of concepts. Much crystallising and making it concrete. Learn this, and learn what it is for. In practice, of course, you learn the new stuff,

	and then relate it back. So mathematics was being taught in an overly linear fashion?
Sheila – fun and confidence	Yes, I think so. It irritates me, this. When they had to learn things. There was no evidence, unless you had very good teachers, of the fun of something. Alright, it's fun to know how nine, times up. You know, 9, 18, 36, 45. [9 = 9 , 1 + 8 = 9 , 3 + 6 = 9 , 4 + 5 = 9] "Oh, I never knew." And when they see this, "oh, yes. It does." And you would know when nine went into a number. That sort of thing. Suddenly going out of the straight and narrow, into investigation.
James	At the weekend, a school teacher showed me how to do the nine times table on my fingers. Do you know that method?
Sheila	Yes. And I have forgotten it. But it is fun, isn't it?
James	It is absolutely amazing.
Sheila	All sorts of things like that get my children, primary school children, interested. And then they get their confidence.
James	Of course it is talking the language of children. Making it a game. Children play. So these mathematical games latch into what children do.
Sheila	Absolutely I was in the hall one day and some children were challenging each other to see if they could jump two steps at a time. One child jumped, "two, four, six," then he said, "it is the same as my two times table." Children are doing well if they can link mathematics to other things.
James	I remember, once, this was when I was quite old, I suppose, a young teenager, working out the power I developed when I jumped up some steps. It's a fairly interesting calculation, involving my weight, the height, the time. I was quite tall. I could jump four or five steps from a standing start. With a stop watch, in a fraction of a second. I was developing something like three and a half kilo Watts

of power. I said, “well, why don’t my muscles burst into flame, then?” It is an enormous amount of power.

Sheila

That is a lovely thing to find out.

James

The answer, of course, is that it lasted such a brief second, that my muscles didn’t get hot enough to burst into flames. There wasn’t that much energy there.

It is interesting, how good demonstrations make mathematics and physics very clear.

Sheila

Absolutely.

The other thing, I don’t know if you found this, I learnt more mathematics when I taught, than I had ever learnt coming up through school and university.

James

You mean you learned it in more depth …

Sheila – influence of learning to teach

My concepts were better formed.

James – universities teach knowledge they have created themselves

Yes, because you have got to teach it. It certainly does refine one’s own concepts when you have got to teach it. And, partly, it is because you are thinking, “How can I communicate this to my pupils or students?” And you deliberately set about building concepts to create a story that you can tell. And that is especially so in universities, where we are teaching new knowledge that we have created ourselves.

Computing is quite good at that. Because computing has to formalise everything that computers are used for, you are used to expressing all sorts of intellectual things as program code. So you have all those skills for developing concepts and presenting them formally.

I am largely self-taught in mathematics. I learnt it by applying the concepts of programming. So I would say, “A proof. Hmm, that is a subroutine. I’ll write myself a subroutine that lets me change this formula into that one. And now I’ll put it together into a bigger one, and a bigger one, and a bigger one.” As a programmer, it gives you a mental structure that some mathematicians lack. The relationship between the text of a proof and how the text can be used to create a proof.

Sheila – computers

I can quite believe it. I was going to say to you. Because I didn't do computing. You see, when new members of staff came into the department I was perfectly prepared to let them do computing. I just did the old fashioned stuff. People said, "you will regret it." And I do regret not learning about computers. But it relieved me enormously, because I had so much administration to do. But as you have been talking, I can see the purpose of mathematics as it has been directed to computing.

James – total mathematics

That's right. Computing creates a new purpose for mathematics. Mathematics was always about solving problems that people had, or giving people techniques to explore new things. Now computers are a thing in themselves and they raise issues that don't arise for human mathematics.

One of those, which the mathematics community has not responded to fully, is the idea of total mathematics. A total system of mathematics must always work. So that the computer always knows what to do next. And a lot of mathematics isn't total. It doesn't work in certain cases. That's O.K. for a human being, they just say, "Oh, well, the technique doesn't work in this case." But a computer can't cope with that. If a technique does not work, it is in big trouble. And so there is a pressure beginning to build, among younger mathematicians, to develop total mathematics. Mathematics that always works. And that raises a lot of issues that haven't been faced very much. People started thinking about it in the 1930s, which is some time ago, but in the mathematical sweep of history, that is fairly recent. So there is a pressure to change mathematics.

Sheila

How far are things with you, for doing what you want to do?

James – support for transreal arithmetic

Well, that's a good question. I have developed this new bit of mathematics. It is a total mathematics. There are other people doing that, too. I am going around publishing papers that this mathematics is useful. I have a very few people who agree with me. There are two people who agree that it is important to computing, it makes computers simpler and faster and safer. So there is a practical side to it. There is one person who agrees that it puts mathematics on a firmer foundation, it deals with the exceptions better. It doesn't overturn any of the existing, positive, results of mathematics, but it gives some new results. And it turns over some of the negative results. Some of the things you couldn't do before, you can do now. And that is always fun. Some of those negative things are to do with when zeros and infinities arise. Physical equations used to break down at infinities, but they don't break down any more. You can still do them. So, for example, you can calculate a power series at an essential singularity, which is not supposed to be possible, but you can just do it. So I am interested in the question of whether that simplifies the modelling of physics. And then the real question is, does the physical world work according to these new number systems? The real world does not seem to get hung up on infinities. Is it because the real world follows some mechanism where infinities don't arise, or is it that the number system that the physical world is operating by, is, in some ways, similar to transreal numbers, rather than ordinary numbers?

And there is one person who thinks transarithmetic might be a better way of teaching arithmetic. So there are, perhaps, four people in the world who agree with me. And a very great many who don't.

Sheila

Well, it's a *start.* It's bound to happen, isn't it? There are people who don't want it to happen.

James – no exceptions

I spoke to another teacher who, after initial resistance, was very positive. And she was positive because suddenly it makes all of arithmetic work. There are no exceptions. You can divide by zero, there are no exceptions. And she was astonished to see that the ordinary rules of arithmetic just continue to work. And she said that pupils would love the fact that things work everywhere.

Sheila

That's good. That's the sort of thing teacher training students like, children as well. Why doesn't it work when I do …

James

Well now it always works.

Sheila

Does it play much on negative numbers?

James – transreal number line and transreal arithmetic

It has no impact on negative numbers. So … I can draw you a picture. Let me switch into talking about transmathematics.

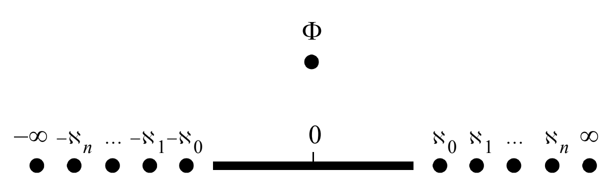

Figure 6.1: The transreal number line with additional infinities

We draw the real-number line through zero, going up through the positive numbers. At the end of the real-number line there is a gap, then there is infinity ∞. On the other side, there are the negative numbers, a gap, and minus infinity. Now this gap explains some of the behaviour of infinity. If you are a real number there is always a bigger real number. So that's an open set. Then there is a gap. Some other infinities drop into the gap, the transfinite numbers, $\aleph_i$. And there is a new number, nullity, Φ that lies off the line. Now infinity is one divided by zero, $\infty = 1/0$. That is axiomatic. It is not in the limit, it is exactly infinity. Minus infinity is minus one over zero, $-\infty = -1/0$. And you might guess what nullity is. It is zero over zero, $\Phi = 0/0$. Now we have defined the infinities and zero over zero as numbers. When you do that, all of the algorithms of arithmetic continue to work. And because nullity is off the number line, you don't get any contradictions.

So I'll show you the rules of arithmetic: a over b, times c over d, equals a time c, all over b times d. In every case. Even when b and d might be zero. This is practically the method you taught in schools. Normally the rule is that b and d musn't be zero, but you rub that rule out and nothing bad happens.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Now let me show you the division rule: a over b, divided by c over d, equals a over b, times d over c. You define division in terms of the reciprocal, not the multiplicative inverse. This works even if any of the numbers are zero.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

Now we do addition. It is a little bit more complicated: a over b, plus c over d, equals a times d, plus b times c, all over b times d. So that is the usual rule, where you have got separate denominators.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

And then there is a special rule: plus or minus one, over zero, plus, plus or minus one, over zero, equals plus or minus one, plus, plus or minus one, all over zero. That's handling the addition of the infinities. And you knew that rule, too. It is the rule for adding numbers with a common denominator.

$$\frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0}$$

Subtraction is just the addition of the opposite of a number. So a over b, minus c over d, equals a over b, plus the opposite of c, that is $-c$, over d.

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{(-c)}{d}$$

Now all of the algorithms of arithmetic work.

Sheila — Isn't that lovely.

James — Now let me ask you, what is three times infinity?

Sheila — It would be infinity.

James – calculating with infinity — So your intuitions tell you that it is infinity. Now imagine an eleven year old child asking, "What is three times infinity?" Well you can show them: three times infinity equals three over one, times one over zero, equals three times one, all over one times zero, equals three over zero.

$$3 \times \infty = \frac{3}{1} \times \frac{1}{0} = \frac{3 \times 1}{1 \times 0} = \frac{3}{0}$$

Sheila — Which is infinity.

James – cancellation — You have to learn the rules for reducing numbers to least terms: any positive number divided by zero is infinity, any negative number divided by zero is minus infinity, and zero divided by zero is nullity. But there is a short cut for transrational numbers. You just cancel the highest common factor, if there is one. So we re-write three as one times three, and re-write zero as zero times three. Now we cancel the common factor, three, and the result is one over zero, which is infinity.

$$\frac{3}{0} = \frac{1 \times 3}{0 \times 3} = \frac{1}{0} = \infty$$

The usual rule would have us cancel the highest common divisor, but that doesn't work, because zero doesn't have a divisor, where divisors are defined via the multiplicative inverse.

Nullity doesn't have a highest common factor, because every integer is a factor of zero, but there is no highest integer.

Some people will grumble about this definition of factors. But their grounds for grumbling are that they don't want to divide by zero.

The last thing you have to watch out for, is always to make the denominator non-negative, and to reduce numbers to least terms before you add, subtract, multiply or divide them.

Sheila

That's lovely. I can see why your teacher liked this. And I can see what you mean about total arithmetic.

James – proofs involving infinity

Now you can calculate things you couldn't before. Let me ask you. Could you prove that infinity is a big number?

Sheila

The only proof I would say, is that there is always one more number.

James

That's the Peano axioms. But the Peano axioms don't deal with infinity. They always say there is a bigger integer than the integer I have now, but they don't say that any integer is infinity.

Let me give you a proof that infinity is big. We will say that a/b is greater than c/d if and only if $(a/b)-(c/d)$ is greater than zero. And I remind you that infinity is bigger than zero. I showed you that in the diagram, and it is an axiom of transarithmetic. All it is saying, is that infinity is positive, so I can distinguish it from negative infinity. Now:

$$\infty - \frac{c}{d} = \frac{1}{0} - \frac{c}{d} = \frac{(1 \times d) - (c \times 0)}{0 \times d} = \frac{1 \times d}{0 \times d} = \frac{1}{0} = \infty > 0$$

So infinity, ∞, is greater than any rational number c/d. You have to do slightly more to show that it works with irrational numbers, and to figure out how infinity relates to itself, to minus infinity, and to nullity. That is the sort of thing you might do in secondary school, or with a very advanced group in primary school, but you can do it using only the rules of arithmetic that you could teach in primary school. No other arithmetic, that children can do, can prove that infinity is big.

Sheila

That's lovely. Yes.

James – preparation for calculus

So in primary school, you can introduce the idea of infinity in a consistent way, using only the arithmetic that pupils are learning anyway, and giving them an absolutely firm foundation for understanding infinity, in calculus, in secondary school.

Sheila – spiral learning of mathematics

What I like about that is that it is always spiral learning in mathematics. You come round to it again and you learn more this time round, and round again. And if your foundations are right you don't have to unlearn anything later, or tell the child that we will deal with that later.

James

Because it is total, you can give a child a complete grasp. Here is the arithmetic, it works for everything. This is infinity. You don't have to put anything off to later.

Sheila

Are you wanting to get this launched as a new project, with examinations?

James – teaching transreal arithmetic in primary and secondary schools

I am interested in the mathematics for itself, and am pushing it in computing. I am keen to help my colleagues, in education, prepare the ground for when they are happy to pursue this mathematics. This mathematics is controversial. Some people don't like the fact that you can divide by zero, and there are other approaches to dividing by zero that some people prefer. None of those approaches is arithmetical. This is the only arithmetical one. So it would be easier to teach this in schools than any of the others, but I don't want to push school teachers faster than they can accept it.

Sheila – golden ages of mathematics

I can remember someone saying that the new mathematics was long overdue. They called the new mathematics, the golden age of mathematics, like the golden ages of the Greeks and Newton.

James – the speed of introduction of new kinds of mathematics

My model of how mathematics gets into schools, is that it gets invented somewhere, often in universities. On average, it takes the inventor half a working life, that is to say, twenty years to come up with the invention. Then it takes twenty years to get a toe hold is some area of university education. At which point, half the university staff, who work for forty years, have learnt the new mathematics and expect it to be taught in secondary schools. And it takes twenty years until half the secondary staff have learnt it. And it takes forty years until all of the primary staff have learnt it. That is a total of one hundred years, which is

	about as long at it took Britain to switch from using Roman numerals to Arabic ones.
Sheila – influences of the second world war	The new mathematics was around in the thirties, but the war had an effect. After the war, people who had been in the RAF, or in gunnery, were streets ahead of everyone else in mathematics. They had learnt such an enormous amount of very practical mathematics.
James	So they taught mathematics in the services?
Sheila – Barnes Wallis	Yes. One of the things I was told, was to look at how arithmetic was taught in the services. Because they couldn't have innumerate soldiers. There was a lot of it done in world war one, too. They didn't mess around. It was very applied.
	My father met Barnes Wallis. His persistence in getting the bouncing bomb was quite remarkable. Mathematics was a matter of life or death.
James	Did you ever meet Barnes Wallis?
Sheila – mathematics teaching has improved	No.
	I think children are taught better now. There is less horror about mathematics. Five years ago, I helped a young girl who had missed out on mathematics because she had been ill for a long time. She was terribly keen and is now at secondary. I asked her how the mathematics is going. She said she had never got stuck and she liked it.
	The practical work took a long time, and this is where things started to get a bad reputation, because the weaker teachers allowed that time to take over and weren't getting down to the more difficult stuff, which they weren't sure of themselves.
James	Well, I think we have covered quite a lot.
Sheila	I think we have covered everything.
James	That has been very helpful. Many thanks.

Conclusion

I had always assumed that the new mathematics was special to Britain, but it seems it was part of a world-wide trend, spurred on by the second world war and the space race. It is interesting that the introduction of the new mathematics was long over-due. Perhaps this is necessary, so that there is a large pool of people who can teach a new mathematics. And it would seem that we must involve universities, teacher trainers, teachers, examiners, government inspectors, and professional associations in the endeavour – with the examination boards playing a key role in speeding the introduction of a new mathematics. All of this will involve huge financial expenditure and will probably only be undertaken when computing or physics makes it obvious that we must be able to divide by zero.

I wonder how long it will take to introduce total mathematics to schools. Can we do anything that will make it less than a century? And when total mathematics is taught in schools, will transarithmetic be a part of it?

CHAPTER 7

What Use is Nullity?

Introduction

I have been asked, by a mathematician, to explain what use nullity is. This strikes me as an odd sort of a question because, surely, anyone who has used nullity knows what use it is, in much the same way that anyone who has used zero knows what use it is. But, for those who have made it this far through the book, without using nullity, I now set out my reasons for believing that nullity is useful. I have tried to structure the list so that it gives broadly useful things before giving specific examples. I hope this will point out the shape of the wood before the reader comes across each tree. The reader who takes the time to re-read the list, many times, will develop a thorough understanding of the neck of the woods I have explored in software, but my advances in the design of computer hardware are too radical to explain in this chapter. I also scatter examples of the mathematical usefulness of nullity around, like acorns on the ground.

I would very much like to hear from anyone else who has found nullity useful, especially if they know of better reasons than I give here. I acknowledge all contributions at the end of the chapter.

Totality

Nullity is the missing link which converts real arithmetic into a total arithmetic. Totality creates new design opportunities that cannot exist in a partial system. Total systems are usually too cumbersome to lift, let alone wield as weapons, but transreal arithmetic is so simple that eleven year old children can learn it, and carry it around in their heads, ready for instinctive use whenever an occasion arises. In time, this will make the whole population better able to design and work with total systems.

Compilers

Any syntactically correct, transreal expression is semantically correct. The compiler need not perform any semantic checks on these expressions, nor need it generate run-time code to perform semantic checks. This makes the compiler simpler to write, smaller, faster and safer. Similarly, the run-time code is simpler, smaller, faster and safer. And the hardware can be made simpler, smaller, faster and safer. When used in a conventional design, these advantages multiply together to give a significant improvement. When used in a radical design their synergies bring about orders of magnitude of improvement.

Programming examples

The following programming examples are written in pseudocode. The reader who cares to try the examples will find that the effort of transcribing them into an existing programming language highlights the computational issues which are the subject of the example. Thus, the reader will be exposed to the issues twice: once when writing the code and again when testing it. In each example, I describe a problem and then solve it using floating-point arithmetic[12] and then transreal arithmetic. No doubt the examples can be re-written more efficiently in any particular programming language. The effort of transforming the programs into their most efficient form will force the reader to consider the issues a third time. It is only by using transreal arithmetic, and comparing it to its alternatives, that one comes to appreciate its strengths and weaknesses.

Membership of a list

Write a program that returns *true* if a given target is a member of a given list and *false* otherwise.

If we use floating-point arithmetic the specification is not clear. Are we to treat *zero* and *minus zero* as the same number? I now decide not. Are we to treat all objects which are not a number as equal? I now decide that we are.

```
define ismember(target, list) -> found;
    false -> found;
    for number in list do
        if   (number = target and 1/number = 1/target)
              or
             (number != number and target != target)
        then true -> found;
        endif;
    endfor;
endefine;
```

If we use transreal arithmetic the specification is clear.

```
define ismember(target, list) -> found;
    false -> found;
    for number in list do
        if  (number = target)
        then true -> found;
        endif;
    endfor;
endefine;
```

Computing the largest element of a list

Return the largest element of a given list.

If we use floating-point arithmetic then the specification is not clear. What size is an object which is not a number? I now decide that it does not have a size. What are we to do if the list is empty or contains only objects that are not a number? I now decide that we are to return a *NaN* in this case.

```
define maximum(list) -> max;
    false -> found;
    -1 / abs(0) -> max;
    for number in list do
         if   number = number and
              number >= max
         then true -> found;
              number -> max;
         endif;
    endfor;

    if   not(found)
    then abs(0) / abs(0) -> max;
    endif;
enddefine;
```

If we use transreal arithmetic then the specification is clear, except that we must decide what to do if the list is empty. I now decide that we are to return *nullity* in this case.

```
define maximum(list) -> max;
    false -> found;
    -1 / 0 -> max;
    for number in list do
         if   number != nullity and
              number >= max
         then true -> found;
              number -> max;
         endif;
    endfor;

    if   not(found)
    then 0 / 0 -> max;
    endif;
enddefine;
```

Computing the supremum of a list

Return the supremum of a list.

If we use floating-point arithmetic then the specification is not clear. What size is an object which is not a number? I now decide that it does not have a size.

```
define supremum(list) -> sup;
    -1 / abs(0) -> sup;
    for number in list do
         if   number = number and
              number > sup
         then number -> sup;
         endif;
    endfor;
enddefine;
```

If we use transreal arithmetic then the specification is clear.

```
define supremum(list) -> sup;
    -1 / 0 -> sup;
    for number in list do
         if   number > sup
         then number -> sup;
         endif;
    endfor;
enddefine;
```

Computing the arithmetic mean of a list

Return the arithmetic mean of a list.

On consulting a textbook of mathematical statistics we read the following definition of the sample mean. (See,[11] pages 254-255).

If $x_1, x_2, \ldots, x_n$ *constitute a random sample, then*

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

is called the sample mean

This is a clear specification. But what are we to do if the list is empty? I now decide that we should return

$$\bar{x} = \frac{0}{0}$$

in this case.

Note that statisticians might take a different view of the appropriate mean of an empty sample.

If we use floating-point arithmetic the specification is now clear, except that we must decide whether or not to raise an exception on division by zero and we must decide which *NaN* to return. I now decide that we should not generate an explicit exception and choose the *NaN* programmatically.

```
define mean(list) -> average;
    abs(0) -> total;
    for number in list do
        total + number -> total
    endfor;
    total / abs(length(list)) -> average;
enddefine;
```

If we use transreal arithmetic then the specification is now clear.

```
define mean(list) -> average;
    0 -> total;
    for number in list do
        total + number -> total
    endfor;
    total / length(list) -> average;
enddefine;
```

Computing the variance of a list

The textbook[11] of mathematical statistics continues as follows.

and

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$$

is called the sample variance.

But suppose that there is just one variate in the sample then $n = 1$ and

$$s^2 = \frac{0}{0}$$

In this case the sample variance is undefined in real arithmetic and the proof that the sample variance is an unbiased estimator of the population variance is faulty in this case. This sort of error, where a corner case is missed, is common in mathematical papers and texts. It is a matter for statisticians (and mathematicians) to decide how to fix this, and similar, errors in their texts.

Now the floating-point specification is clear when we choose which *NaN* to return with no explicit handling of exceptions.

```
define variance(list) -> var;
    mean(list) -> m;
    abs(0) -> total;
    for number in list do
        total + (number - m)**2 -> total
    endfor;
    total / abs(length(list)) -> var;
enddefine;
```

But the specification is faulty for transreal arithmetic. The term $x_i - \bar{x}$ is used to return the signed distance from $\bar{x}$ to x. This quantity[7] must be computed explicitly in transreal arithmetic.

```
define signed_distance(a, b) -> distance;
    if   a = b
    then 0 -> distance
    else a - b -> distance
    endif;
enddefine;
```

Now we have:

```
define variance(list) -> var;
    mean(list) -> m;
    0 -> total;
    for number in list do
        total + signed_distance(number, m)**2 -> total
    endfor;
    total / length(list) -> var;
enddefine;
```

Conclusion

The above examples expose some of the strengths and weaknesses of the transreal number system when compared to real arithmetic and floating-point arithmetic. Real arithmetic is the easiest to use, but it is partial and does not apply to some calculations which arise naturally, and explicitly, in computational problems. Both floating-point and transreal arithmetic are total and apply to all computational problems (of a finite length), but transreal arithmetic is often more concise and is always better specified.

The reader is encouraged to try many more examples. It is only through repeated use of competing systems that one can decide which is more useful. For my part, I maintain that transreal arithmetic preserves more information about the sign and magnitude of numbers than any total arithmetic that can be taught successfully to school children. As transreal arithmetic uses only algorithms which are already taught in schools, it is well adapted to presentation in schools. I also maintain that transreal arithmetic has the potential to solve physical problems that are inaccessible to real arithmetic, but I have proposed only one example of this.[7] I maintain that transreal arithmetic has commercial advantages for handling exceptions in floating-point arithmetic. In time, it will become blatantly obvious whether, or not, transreal arithmetic is useful. In the mean time, there is no alternative to trying it in practical, theoretical and pedagogic settings.

Acknowledgement

Steve Leach, developer of the compiler for the Perspex chip, suggested an algorithm for computing the arithmetic mean of a list of numbers and for searching a list for a specific or a largest element.

Contribution from Steve Leach

Programming is the art of putting simple functions together to create complex and useful ones. But functions do not always work together well. Experience shows us that programmers focus on getting the error-free case correct, but occasionally mishandle the alternative cases where something goes wrong. Throwing exceptions does not improve matters because, while it does simplify the coding of the main error-free cases, it does not pinpoint the faults in the alternative, erroneous, cases. Rather, it gives up on them in an all or nothing way.

It is always better to program with well-chosen functions whose inputs and outputs match up. But for computer arithmetic (and it is hard to think of a more basic or important set of functions) this is easier to say than do, because the mathematics which it approximates lacks total functions. This is why a theory of total arithmetic is important.

Having a total arithmetic makes it possible to compose our arithmetical functions together into large, complex functions simply by connecting the inputs and outputs together. We do not need exception handling code to make this work, nor code to check that the outputs of one function are in the acceptable range of another function.

The use of total functions combines especially well with static type-checking. Once a function is type checked it will run without generating any run-time exception.

Building on this idea, it would be possible for a type system to perform a dimensional analysis of the physical or computational units that are assigned to numbers. This would prevent satellites from crashing into Mars when accelerations in a foot-pound-second system are converted to SI units of kilogram-metre-second, and it would prevent invoices in US dollars being paid in Hong Kong dollars.

At present, type systems can handle simple conversions by providing type aliases for numbers, but they cannot check general arithmetical calculations, because these are stated in dimensionless numbers. It is astonishing that no mainstream computer language currently supports dimensional analysis, which is a technique used every day by engineers and scientists around the world.

Transreal arithmetic is a better foundation for computer arithmetic, maximising the benefits of type analysis and making programming easier, safer and more efficient.

APPENDIX 1

Answers

4 The Graph of the Reciprocal

4.1 There are many possible answers. Suppose that $\perp = n/0$ for all numbers, n. Let us use nullity to model bottom. We define a function, f, as:

$$f(x) = \begin{cases} \Phi & : x \in \{-\infty, \infty\} \\ x & : \text{otherwise} \end{cases}$$

Now, instead of operating just on some definition of the reciprocal, $r(x)$, we operate on $f(r(x))$. And we choose r so that it gives whatever behaviour we want at the extreme left and right of the graph of the reciprocal.

4.2 There are many possible answers. Let us use positive infinity to model an unsigned infinity. We define a function, f, as:

$$f(x) = \begin{cases} \infty & : x = -\infty \\ x & : \text{otherwise} \end{cases}$$

Now, instead of operating just on some definition of the reciprocal, $r(x)$, we operate on $f(r(x))$. And we choose r so that it gives whatever behaviour we want in the graph of the reciprocal.

4.3 The argument presented is a slightly edited version appearing in an article in Wikipedia, a web site that aims to provide encyclopaedic synopses of accepted knowledge. It is not clear from the argument what the accepted knowledge is supposed to be, but on analysing the argument, using the logic of real arguments,[10] we come to the conclusion that it is an argument that calculus cannot specify division by zero. Nonetheless, it is our experience that the arguments presented in the article are used by some people to attempt to show that division by zero is impossible. Such attempts are doomed to failure because arithmetic is logically prior to calculus, so that if any contradiction were found it would imply a fault in calculus, not arithmetic. Secondly, calculus deals with division by non-zero, infinitessimal numbers so it has nothing to say about division by zero.

The first thing in the argument is a diagram. We examine the diagram and find that it is unlabelled where the graph approaches the axes. Therefore, we cannot tell if the graph approaches or arrives at: zero, an unsigned infinity, one or more signed infinities, or bottom. We cannot tell if the graph is connected.

The graph is not an accurate rendition of the reciprocal so we suppose it is a sketch. Whenever we are unsure of anything we follow the *Principle of Charity*[10] and assume the best interpretation we can find. In doing this, we try to squeeze as much information out of an argument as possible. We are not interested in scoring points off the author. On checking the original source we find that the diagram there is accurate so the diagram in the present chapter has been edited from the original. In fact, I drew the diagram this way because it is less labour than preparing an accurate drawing. Nothing is lost to this device and something is gained. The sketch preserves all of the information that is needed for the purposes of the argument, and it does so in a more exaggerated, and therefore clearer, picture.

The figure legend is clearly supplied by the author of the present chapter, not the author of the argument.

> *At first glance it seems possible to define $a/0$ by considering the limit a/b as b approaches zero.*

Why does the author say "at first glance?" Is this introducing a mathematical hypothesis or is it a piece of propaganda to encourage us only to "glance" at the proposition and not to think about it deeply? The proposition is:

> *It seems possible to define $a/0$ by considering the limit a/b as b approaches zero.*

We are not interested in the meta-level reasoning of "seeming possibility" so we translate the proposition into a concrete one:

> *It is possible to define $a/0$ by considering the limit a/b as b approaches zero.*

Why should we believe this proposition? The limit is taken in the approach to zero, not exactly at zero, so why should the limit have anything to say about $a/0$? I maintain that it has nothing to say in this case.

> *For any positive a it is known that:*

$$\lim_{b \to 0^+} \frac{a}{b} = +\infty$$

This is true. But is it relevant?

> *For any negative a it is known that:*

$$\lim_{b \to 0^+} \frac{a}{b} = -\infty$$

Again, this is true. But is it relevant?

> *Therefore $a/0$ is defined to be $+\infty$ when a is positive and as $-\infty$ when a is negative.*

What justifies the logical connective "therefore?" I do not believe that this argument from limits is relevant, but I understand that the author of the Wikipedia article is introducing a hypothesis that the definition is consistent with what is known about limits. If anything turned on this interpretation, I might attempt to contact the author to clarify what his or her intention is.

However, taking the limit from the right is arbitrary.

No. There are a great many technical requirements on the taking of a limit. Some of these specify in which direction the limit can be taken in particular cases.

The limits could be taken from the left.

This is either false or else it exposes an unstated assumption that the graph of the reciprocal is connected at $x = 0$. Henceforth, we hold the most generous hypothesis, viz, the graph is connected at $x = 0$.

In which case $a/0$ is defined to be $-\infty$ when a is positive and as $+\infty$ when a is negative.

This is true, but it is a tautology. It follows from the unstated assumption that the graph is connected at $x = 0$ so that $-\infty = \infty$.

This can be further illustrated using the following equation (when it is assumed that several properties of the real numbers apply to the infinities)

$$+\infty = \frac{1}{0} = \frac{1}{-0} = -\frac{1}{0} = -\infty$$

What properties of the real numbers are we supposed to assume apply to the infinities? What could possibly justify us in that belief? I do not believe there is any property of the real numbers that can justify:

$$\frac{1}{-0} = -\frac{1}{0}$$

In this context, I am not even sure what is meant by:

$$-\frac{1}{0}$$

Does it mean the same as one or more of the following:

$$\frac{1}{-0} \text{ or } \frac{-1}{0}$$

Whatever it may mean, it is not consistent with transreal arithmetic. If we run through the equation, reducing fractions to canonical form, we obtain the following harmless tautology:

$$+\infty = \frac{1}{0} = \frac{1}{-0} = \frac{1}{0} = +\infty$$

Returning to the argument, we read:

> *Which leads to $+\infty = -\infty$ which would be a contradiction with the standard definition of the extended real-number line.*

The equation does not lead to a contradiction in transreal arithmetic, but let us suppose that it does lead to a contradiction in the author's unstated generalisation to an extended-real arithmetic. But on this reading, discussion of the extended real-number line is irrelevant, because of the unstated assumption that the graph is connected at $x = 0$.

> *Furthermore there is no obvious definition of $0/0$ that can be derived by considering the limit of a ratio.*

Further to what? Further to an irrelevant observation? I take the author of the Wikipedia article to say:

> *There is no obvious definition of $0/0$ that can be derived by considering the limit of a ratio.*

Why does the author say "obvious." What difference would it make to us if the definition were obvious or so unobvious as to be a stroke of genius? Regardless of this, we hold that no definition of $0/0$ can be derived by considering the limit of a ratio, because these limits are taken in the neighbourhood of zero, not exactly at zero.

> *The limit:*
>
> $$\lim_{(a, b) \to (0, 0)} \frac{a}{b}$$
>
> *does not exist.*

This is true when a, b range over the real numbers. It is not true, for example, when a, b are constant zero. But is it relevant?

> *Limits of the form*
>
> $$\lim_{x \to 0} \frac{f(x)}{g(x)}$$
>
> *in which both of $f(x)$ and $g(x)$ approach zero, as x approaches zero, may equal any real or infinite value, or may not exist at all, depending on the particular functions f and g.*

True. But is it relevant?

> *These and other, similar, facts show that $0/0$ cannot be well defined as a limit.*

What are the similar facts? At best the above argument shows that $0/0$ cannot be obtained as the limit, ranging over some non-constant values, of a ratio. But this is obvious to us and is irrelevant to the question of whether or not $0/0$ can be a limit of some function. In fact it can be so obtained, as a future publication will show.

Having examined the content of the argument we now re-read it to see what its claims are. What does the argument set out to prove? We cannot tell, this is not stated. What conclusion is drawn? We cannot tell, there is no written conclusion to the argument. At best this is a summary of approaches to division by zero which demonstrates that calculus has nothing to say on this topic.

Now we re-read the original source and find references to other works. Perhaps these will provide useful information?

Finally, I make a note of where the fallacious argument was published. It is a useful debating device to point out that division by zero is not widely understood. This demonstrates the academic need for a clear account of division by zero and goes some way to demonstrating to a reviewer of my work that I have considered the views of others. Of course, I might look out for more authoritative criticisms, and use these to bolster my claims that division by zero is not widely understood, and that transreal arithmetic provides a very good account of it.

APPENDIX 2

Notation

Strictly transreal numbers

The strictly transreal numbers are: negative infinity $-\infty = (-1)/0$; positive infinity $\infty = 1/0$; and nullity $\Phi = 0/0$.

Addition, subtraction, multiplication, division

$+, -, \times, \div$ are, respectively, the operations of addition, subtraction, multiplication, and division. They apply to ordinary numbers in the ordinary way, but also apply to the strictly transreal numbers.

Ordinary multiplicative inverse

In ordinary mathematics, a^{-1}, is known as the *multiplicative inverse* and we have $a \times a^{-1} = \frac{a}{a} = 1$ when a is not zero. It comes as a shock to many people to discover that the multiplicative inverse is not the whole of division.

Transreciprocal

$\left(\frac{n}{d}\right)^{-1} = \frac{d}{n}$ In transarithmetic, the superscript minus one, -1, denotes the transreciprocal as shown. This includes the ordinary reciprocal, which is defined via the multiplicative inverse. The *transreciprocal* also applies to the strictly transreal numbers which have no multiplicative inverse.

Parentheses

Parentheses, round brackets, are evaluated, as usual, from the innermost bracket to the outermost. The result is then written without brackets. For example, $(2 \times (4 - 3)) = (2 \times 1) = 2$, and $((2 \times 4) - 3) = (8 - 3) = 5$. Parentheses can be

	used to distinguish the negation of a single number from a subtraction of two numbers. Thus, $3 + (-2) = 3 - 2 = 1$ and $(2 \times (3 - 4)) = (2 \times (-1)) = -2$.
Equals, not equals	$=$, $\neq$ are the operations equals and not-equals.
Not	$\neg a$ is true when a is false, and is false when a is true. The symbol "$\neg$" is known as "not." For example, $2 \neq 3$ means that two is not equal to three, which is a true statement, and $\neg(2 = 3)$ means the same thing, that two is not equal to three.
Less than, less than or equals, greater than, greater than or equals	$a < b$, $a \leq b$, $a > b$, $a \geq b$ mean, respectively, a is less than b; a is less than or equal to b; a is greater than b; a is greater than or equal to b.
Not less than, not greater than	$a \nless b$ means that a is not less than b. This may be because $a \geq b$ or $a = \Phi$. Similarly $a \ngtr b$ means that a is not greater than b. This may be because $a \leq b$ or $a = \Phi$.
Alternatives introduced by comma	The comma, "," introduces an alternative. For example, $a \neq -\infty, \Phi$, means, "a is not equal to negative infinity and a is not equal to nullity."
Alternatives introduced by plus and minus	$\pm a$ introduce the alternatives, $+a$ and $-a$.
When	The colon, ":" means "when" or "such that." For example, $a + \infty = \infty : a \neq -\infty, \Phi$ means, "a plus infinity equals infinity, when a is not equal to negative infinity or nullity."
If then	$a \Rightarrow b$ means that if a is true then b is true. It is also read as, "a implies b."
If and only if	$a \Leftrightarrow b$ means, "if a is true then b is true and if b is true then a is true". This is also read as, "a is true if and only if b is true."
There exists an	$\exists a$ means, "there exists an a."
For all	$\forall a$ means, "for all a."
And	$a \wedge b$ is true when both of a, b are true, and is false when either or both of a, b are false. This is read as "a and b."
Or	$a \vee b$ is true when either or both of a, b are true, and is false when both of a, b are false. This is read as "a or b."

Sign The function $\operatorname{sgn}(a)$ is used as a shorthand so that $\operatorname{sgn}(a) = -1$ when $a < 0$, $\operatorname{sgn}(a) = 0$ when $a = 0$, $\operatorname{sgn}(a) = 1$ when $a > 0$, and $\operatorname{sgn}(a) = \Phi$ when $a = \Phi$.

APPENDIX 3

Annotated Bibliography

History of zero

1 Anonymous. *A history of zero* at http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Zero.html accessed on 14 February, 2008.

This article on the history of zero is clearly written with a very light use of mathematical formulae. It has good cross-references to other web-based material, but has very limited citations to paper texts.

Division by zero

2 Anonymous. *Division by zero* at http://en.wikipedia.org/wiki/Division_by_zero accessed on 7 March, 2008.

This article, intended for the general reader, reviews various mathematical facts relating to division by zero. It is an excellent source of fallacies.

James Anderson

3 Anonymous. *James Anderson (Computer Scientist)* at http://en.wikipedia.org/wiki/James_Anderson_(computer_scientist) Accessed on 2 April 2009.

This Wikipedia article presents some of my biography and describes the events surrounding stories by the BBC on my work. It also gives a summary of my work and relates it to other works. The biographical details are correct, but the technical summaries are faulty. Wikipedia's rules prevent me from correcting an article on myself and no Wikipedia author has checked facts with me.

Nonetheless the errors in the article are very gradually being weeded out by Wikipedia authors.

First paper introducing the point at nullity and perspexes

4 J. A. D. W. Anderson, "Representing geometrical knowledge," in Phil. Trans. R. Soc. Lond., series B, vol. 352, no. 1358, pp. 1129 - 1139.

This paper introduces the point at nullity via a geometrical construction in perspective space. Some calculations in projective geometry are made total by inclusion of this point, but the resulting geometry is not arithmetised. This paper also introduces the perspective simplex (perspex) which describes geometrical shapes and transformations. The goal of developing a logical perspex which can be used to programme a computer is made explicit.

Arithmetisation of transreal numbers

5 J. A. D. W. Anderson, "Perspex Machine VII: The Universal Perspex Machine" in Vision Geometry XIV, Longin Jan Lateki, David M. Mount, Angela Y. Wu, Editors, Proceedings of SPIE Vol. 6066 (2006).

This computational article describes transreal arithmetic in terms of extended operations of rational arithmetic. It is the first occasion on which it is proposed to teach transreal arithmetic in schools. It is argued that nullity is a number because it is a solution to a trigonometric equation. It is argued that the Turing machine is inherently spatial because Turing states that the machine's symbols are compact spaces. The Perspex machine is simplified so that it operates on general-linear transformations, not the more complicated perspective transformations of earlier versions of the machine. It is argued that the Perspex machine can compute with any badly formed formulae. A number of philosophical issues are dealt with and *NaN* is criticised. The chapter on *NaN* in the present book contains a more incisive criticism of *NaN,* though it does not reproduce all of the arguments in the paper.

Axioms of transreal numbers

6 J. A. D. W. Anderson, Norbert Völker, Andrew A. Adams "Perspex Machine VIII: Axioms of Transreal Arithmetic" in Vision Geometry XV, Longin Jan Lateki, David M. Mount, Angela Y. Wu, Editors, Proceedings of SPIE Vol. 6499 (2007).

This mathematical article presents the axioms of transreal arithmetic and some notation for referring to various sets of transnumbers. It describes how the axioms are translated into higher order logic and are proved consistent in a computer proof system. There is no description of the proof itself, though the

paper contains a pointer to an on-line version of the proof. Many elementary theorems derived by the computer proof system are also given.

Topology of transreal numbers

7 J. A. D. W. Anderson, "Perspex Machine XI: Topology of the Transreal Numbers" in IMECS 2008, S.I. Ao, Oscar Castillo, Craig Douglas, David Dagan Feng, Jeong-A Lee Editors, Hong Kong, pp. 330-338, March (2008).

This mathematical article presents the topology of the transreal numbers and shows how the two's complement encoding of numbers describes transintegers better than integers. This is the first paper in which the signed infinities are taken to be disconnected from the real number line.

New mathematics

8 E.Biggs & S. Roberts, *Teaching Primary Mathematics,* Holmes McDougal, Edinburgh, 1986.

This book, for primary school teachers, presents practical examples and teaching methods used in the new mathematics.

Wheels

9 J. Carlström: "Wheels — on division by zero" *Mathematical Structures in Computer Science,* 14(2004): no. 1, 143-184. Also available at http://www.math.su.se/~jesper/research/wheels on 14 February, 2008.

This mathematical article deals with division by an element zero of a ring. The generalisation of division is perfectly natural from an algebraic point of view, but does not preserve the maximum possible information about magnitude and sign when applied to real numbers. The arithmetic of wheels is quite different from transarithmetic. For example, the distributivity laws are different in the two approaches.

Logic of real arguments

10 A. Fisher *The Logic of Real Arguments,* Cambridge University Press, 1988.

This book encourages the general reader to rely on expert opinion less, by assessing arguments, written in English, for themselves. The reader is shown a three step procedure for assessing arguments. Firstly, the argument is identified by looking for linguistic cues indicating inferences, and by supplying any missing inferences that are needed to make a good argument. Secondly, the inferential pathways through the argument are recorded in a diagram or in a linear, textual form. Finally, the argument is tested by applying the *criterion of assertabiliy* – judging by appropriate standards of evidence or

appropriate standards of what is possible, could the premises be true and the conclusion false? If so, the argument is unsound, otherwise it is sound.

The book promotes the *Principle of Charity,* that one reads the best argument into a text that it can bear, or else treats the text as not presenting any argument if no good argument can be read into it. The motive for this is to extract as much useful information from the text as possible. Analysing arguments is not to be taken as an exercise in scoring points off the author.

The methods in the book can be used in reverse – to test one's own written arguments and improve their written expression.

The book deals with various topics in the social sciences and has chapters on accessing scientific arguments, the philosophical basis of the method, and an introduction to formal logic.

Mathematical Statistics

11 J. E. Freund & R.E. Walpole, *Mathematical Statistics,* Prentice Hall, Englewood Cliffs, New Jersey, USA, 3rd edn. 1980, originally 1962.

This text book introduces mathematical statistics. It has some discussion of practical, statistical tests.

NaN

12 *IEEE Standard 754 for Binary Floating-Point Arithmetic* (ANSI/IEEE Std 754-1985).

This international standard describes the layout of bit patterns in floating-point numbers. The standard does not use a formal notation checked by computer and, consequently, contains some ambiguities. The computing industry has, however, resolved these ambiguities by maintaining hardware designs that are compatible with early reference hardware. Thus, the standard can be read in a consistent way. The standard defines bit patterns that denote objects that are *N*ot *a N*umber, *NaN*. It is defined that $NaN \neq NaN$ for all of these objects. I hold that this is dangerous because it breaks a cultural stereotype amongst mathematicians, computer programmers, and the general public, in that they expect any object to be equal to itself. I propose that it would be safer to replace *NaN* with nullity because nullity adheres to this cultural stereotype. The arithmetical properties of nullity have been formally checked by computer.

Bottom type, programming languages

13 B.C. Pierce, *Types and Programming Languages,* MIT Press, Cambridge Massachusetts, 2002.

This text book describes how to use type theory to design safe, computer languages. The structure of a languages is described formally by a recursive grammar and its meaning is described formally by an operational semantics. The operational semantics describes how terms in the grammar change the sate of an abstract machine. High levels of abstraction are used to analyse the properties of a language and low levels are used to implement it. The book is supported by on-line resources.

The book describes the bottom type as an empty type, with no elements, that is a subtype of every type. This is analogous to the empty set which has no elements and is a subset of every set. The bottom type has several uses. For example, a function whose return type is the bottom type does not return to its caller.

Type theory cannot perform compile time checks on division by zero using ordinary number systems. I note that no such check is required if transreal arithmetic is used so a type system can cope with division by zero. The use of transreal arithmetic would make typed languages even safer.

Bottom element, Denotational Semantics

14 D. Scott, "Data Types as Lattices" in *Proceedings of the International Summer Institute and Logic Colloquium, Kiel,* published as *Lecture Notes in Mathematics,* Springer, Berlin, vol 499, pp. 579-651, (1975). There is a paper by the same name in SIAM J. Comput. Vol. 5, Issue 3, pp. 522-587 (1976).

The book chapter is a mathematical paper which describes computation in terms of the power set of non-negative integers. In this model, bottom is the empty set. Bottom may be used to denote that the result of a computation is undefined. The book chapter does not have a bibliography.

Ordinary calculus

15 M. Spivak, *Calculus,* W.A. Benjamin, London, (1967).

This text book provides a very clear introduction to calculus. It uses many sketches of graphs to present technical points in an easy way. It gives both examples and counter-examples of continuity, differentiability, and the like. It discusses the more common notations for derivatives.

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Subjects

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Authors and Historical Figures