DRAFT: Transnumber Derivations and Model

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Contents

1 Transarithmetic: Axiomatic Classes

theory *TransNumberAxclass*

imports *Main Real*

begin

1.1 New constants: xor, infinity, sgn

Note: This "list" xor is different from binary xor when there are three or more arguments.

consts
  $xor :: bool\ list \Rightarrow bool$
primrec
  $xor\ [] = False$
  $xor\ (a \# x) = ((xor\ x \wedge \neg\ a) \mid (\neg\ (list\text{-}ex\ id\ x) \wedge a))$

lemma *xor-singleton*: $xor\ [b] = b$
  ⟨*proof*⟩

lemma *xor-pair*: $xor\ [a,b] = (a \neq b)$
  ⟨*proof*⟩

lemma *xor-triple*:
  $xor\ [a,b,c] = ((a \wedge \neg\ b \wedge \neg c) \vee (\neg\ a \wedge b \wedge \neg c) \vee (\neg\ a \wedge \neg\ b \wedge c))$
  ⟨*proof*⟩

axclass *infinity* $<$ *type*

Note: there is no separate constant minus-infinity

consts
  $infinity :: 'a::infinity\ (\infty\ 100)$

axclass *nullity* $<$ *type*
  consts
    $nullity:: 'a\ ::nullity\ (\Phi)$

axclass *sgn* $<$ *zero*,*one*,*minus*, *ord*

consts
  $sgn :: 'a :: sgn \Rightarrow 'a$

axclass *trans-sgn* $<$ *sgn*, *nullity*
  trans-sgn:
  $sgn\ a = (if\ 0 < a\ then\ 1\ else$
    $if\ 0 = a\ then\ 0\ else$
    $if\ a < 0\ then\ -\ 1\ else$
    $(*\ a = \Phi\ *)\quad \Phi)$

1.2 Axiomatic class trans_add

axclass

trans-add $<$ *zero*, *infinity*, *nullity*, *plus*, *minus*
A1: $a + (b + c) = (a + b) + c$
A2: $a + b = b + a$
A3: $0 + a = a$
A4: $\Phi + a = \Phi$
A5: $[\![a \neq -\infty; a \neq \Phi]\!] \Longrightarrow \infty + a = \infty$

A6: $a - b = a + (-b)$
A7: $-(-a) = a$
A8: $[\![a \neq \infty; a \neq -\infty; a \neq \Phi]\!] \Longrightarrow a - a = 0$
A9: $-\Phi = \Phi$
A10: $[\![a \neq \infty; a \neq \Phi]\!] \Longrightarrow a - \infty = -\infty$
A11: $\infty - \infty = \Phi$

instance *trans-add* $\subseteq$ *comm-monoid-add*
$\langle proof \rangle$

1.3 Axiomatic class trans_mult (Axioms A12-A28)

axclass
trans-mult $<$ *trans-add*, *trans-sgn*, *one*, *times*, *inverse*, *ord*

A12: $a * (b * c) = (a * b) * c$
A13: $a * b = b * a$
A14: $1 * a = a$

A15: $\Phi * a = \Phi$
A16: $\infty * 0 = \Phi$
A17: $a / b = a * \mathit{inverse}\ b$
A18: $[\![a \neq 0; a \neq \infty; a \neq -\infty; a \neq \Phi]\!] \Longrightarrow a / a = 1$
A19: $a \neq -\infty \Longrightarrow \mathit{inverse}\ (\mathit{inverse}\ a) = a$
A20: $\mathit{inverse}\ 0 = \infty$
A21: $\mathit{inverse}\ (-\infty) = 0$
A22: $\mathit{inverse}\ \Phi = \Phi$

A23: $(\infty * a = \infty) = (0 < a)$
A24: $(\infty * a = -\infty) = (a < 0)$
A25: $0 < \infty$

A26: $(0 < a - b) = (b < a)$
A27: $(a > b) = (b < a)$
A28: $\mathit{xor}\ [a < 0, a = 0, 0 < a, a = \Phi]$

A29: $\neg ((a = \infty \lor a = -\infty) \land \mathit{sgn}\ b \neq \mathit{sgn}\ c \land (b + c \notin \{0,\Phi\}))$
$\Longrightarrow a * (b+c) = (a * b)+(a * c)$

A30: $a \leq b = (a = b \lor a < b)$

instance *trans-mult* $\subseteq$ *comm-monoid-mult*
$\langle proof \rangle$

1.4 Axiomatic classes trans_complete and trans_reals

constdefs
lattice-complete :: $('a::ord)$ *set* $\Rightarrow$ *bool*
lattice-complete *xs* ==
$\forall\ ys.\ ys \subseteq xs \longrightarrow (\exists\ u \in xs.\ (\forall\ y \in ys.\ y \leq u)$
$\wedge\ (\forall\ v \in xs.\ (\forall\ y \in ys.\ y \leq v) \longrightarrow u \leq v))$

axclass *trans-complete* < *trans-add*, *one*, *times*, *inverse*, *ord*
A31: *lattice-complete* $\{x.\ x \neq \Phi\}$

axclass *trans-reals* < *trans-mult*, *trans-complete*

TODO: validate definition by proving lattice-completeness of [0..1]

end

2 Transarithmetic Theorems

theory *TransNumberDerivations*

imports *TransNumberAxclass*

begin

declare *A3*[*simp*] *A4*[*simp*] *A7*[*simp*] *A9*[*simp*] *A11*[*simp*]
A14[*simp*] *A15*[*simp*] *A16*[*simp*]
A20[*simp*] *A21*[*simp*] *A22*[*simp*] *A25*[*simp*]

constdefs
reals :: $('a::trans\text{-}reals)$ *set*
reals == $\{x.\ x \neq \Phi \wedge x \neq \infty \wedge x \neq -\infty\}$

Note : following subclassing allows reuse of standard add_ac, etc.

instance *trans-reals* $\subseteq$ *comm-monoid-add* $\langle proof \rangle$

2.1 Very elementary equations

lemma *additive-identity-right*[*simp*]: $(x :: 'a :: trans\text{-}reals) + 0 = x$
$\langle proof \rangle$

lemma *multiplicative-identity-right*[*simp*]: $(x :: 'a :: trans\text{-}reals) * 1 = x$
$\langle proof \rangle$

lemma *additive-nullity-right*[*simp*]: $(x :: 'a :: trans\text{-}reals) + \Phi = \Phi$
$\langle proof \rangle$

lemma *additive-infinity-right*: $[\![x \neq -\infty; x \neq \Phi]\!] \Longrightarrow (x :: 'a :: trans\text{-}reals) + \infty = \infty$
$\langle proof \rangle$

lemma *minus-minus*[*simp*]: $(x :: 'a :: trans\text{-}reals) - - y = x + y$
$\langle proof \rangle$

lemma *zero-mult-infinity*[*simp*]:
$0 * (\infty :: 'a :: trans\text{-}reals) = \Phi$
$\langle proof \rangle$

lemma *nullity-minus-left*[*simp*]: $\Phi - (x :: 'a :: trans\text{-}reals) = \Phi$
$\langle proof \rangle$

lemma *nullity-minus-right*[*simp*]: $(x :: 'a :: trans\text{-}reals) - \Phi = \Phi$
$\langle proof \rangle$

lemma *zero-minus-eq-uminus*[*simp*]: $(0 :: 'a :: trans\text{-}reals) - x = -x$
$\langle proof \rangle$

lemma *uminus-eq-uminus*[*simp*]: $(-(x :: 'a :: trans\text{-}reals) = -y) = (x = y)$
$\langle proof \rangle$

lemma *uminus-add-eq-minus*: $-x + y = y - (x :: 'a :: trans\text{-}reals)$
$\langle proof \rangle$

lemma *uminus-minus-commute*: $-x - y = - y - (x :: 'a :: trans\text{-}reals)$
$\langle proof \rangle$

lemma *x-add-y-minus-y*:
$[\![y \neq \Phi; y \neq \infty; y \neq -\infty]\!] \Longrightarrow x + y - y = (x :: 'a :: trans\text{-}reals)$
$\langle proof \rangle$

lemma *uminus-x-add-x*: $[\![x \neq \infty; x \neq -\infty; x \neq \Phi]\!] \Longrightarrow -x + x = (0 :: 'a :: trans\text{-}reals)$

$\langle proof \rangle$

lemma *x-minus-y-add-y*:
$[\![y \neq \Phi; y \neq \infty; y \neq -\infty]\!] \Longrightarrow x - y + y = (x::'a::trans\text{-}reals)$
$\langle proof \rangle$

lemma *uminus-eq-nullity-iff* [*simp*]: !! $x::'a::trans\text{-}reals.$ $(-x = \Phi) = (x = \Phi)$
$\langle proof \rangle$

lemma *uminus-eq-infinity-iff* [*simp*]: !! $x::'a::trans\text{-}reals.$ $(-x = \infty) = (x = -\infty)$
$\langle proof \rangle$

lemma *infinity-minus*: $[\![x \neq \Phi; x \neq \infty]\!] \Longrightarrow \infty - x = (\infty::'a::trans\text{-}reals)$
$\langle proof \rangle$

2.2 Distinctness of six basic constants: 0, nullity, +1,-1, plus/minus infinity

lemma *not-zero-less-zero*[*simp*]: $\neg\ 0 < (0::'a::trans\text{-}reals)$
$\langle proof \rangle$

lemma *zero-noteq-nullity*[*simp*]: $(0\ ::'a::trans\text{-}reals) \neq \Phi$
$\langle proof \rangle$

lemmas *nullity-noteq-zero*[*simp*] = *zero-noteq-nullity*[*THEN not-sym*]

lemma *zero-noteq-infinity*[*simp*]: $(0\ ::'a::trans\text{-}reals) \neq \infty$
$\langle proof \rangle$

lemmas *infinity-noteq-zero*[*simp*] = *zero-noteq-infinity*[*THEN not-sym*]

lemma *nullity-not-less-zero*[*simp*]: $\neg\ \Phi < (0\ ::'a::trans\text{-}reals)$
$\langle proof \rangle$

lemma *zero-not-less-nullity*[*simp*]: $\neg\ 0 < (\Phi\ ::'a::trans\text{-}reals)$
$\langle proof \rangle$

lemma *infinity-noteq-nullity*[*simp*]: $(\infty::'a::trans\text{-}reals) \neq \Phi$
$\langle proof \rangle$

lemmas *nullity-noteq-infinity*[*simp*] = *infinity-noteq-nullity*[*THEN not-sym*]

lemma *zero-less-one*[*simp*]: $(0\ ::'a::trans\text{-}reals) < 1$
$\langle proof \rangle$

lemma *not-one-less-zero*[*simp*]: $\neg\ (1 < (0::'a::trans\text{-}reals))$
$\langle proof \rangle$

lemma *zero-noteq-one*[*simp*]: $(0\ ::'a::trans\text{-}reals) \neq 1$

⟨*proof*⟩

lemmas *one-noteq-zero*[*simp*] = *zero-noteq-one*[*THEN not-sym*]

lemma *one-noteq-infinity*[*simp*]: $(1 :: 'a::trans\text{-}reals) \neq \infty$
⟨*proof*⟩

lemmas *infinity-noteq-one*[*simp*] = *one-noteq-infinity* [*THEN not-sym*]

lemma *nullity-noteq-one*[*simp*]: $\Phi \neq (1 :: 'a::trans\text{-}reals)$
⟨*proof*⟩

lemmas *one-not-nullity* [*simp*] = *nullity-noteq-one* [*THEN not-sym*]

lemma *minus-one-less-zero*[*simp*]: $(- 1 :: 'a::trans\text{-}reals) < 0$
⟨*proof*⟩

lemma *minus-one-mult-infinity*[*simp*]: $(- 1 :: 'a::trans\text{-}reals) * \infty = - \infty$
⟨*proof*⟩

lemma *zero-noteq-minus-one*[*simp*]: $(0::'a::trans\text{-}reals) \neq - 1$
⟨*proof*⟩

lemmas *minus-one-noteq-zero*[*simp*] = *zero-noteq-minus-one* [*THEN not-sym*]

lemma *not-one-less-minus-one*[*simp*]: $\neg ((0::'a::trans\text{-}reals) < - 1)$
⟨*proof*⟩

lemma *one-noteq-minus-one*[*simp*]: $(1::'a::trans\text{-}reals) \neq - 1$
⟨*proof*⟩

lemmas *minus-one-noteq-one*[*simp*] = *one-noteq-minus-one* [*THEN not-sym*]

lemma *infinity-noteq-minus-one*[*simp*]: $\infty \neq (- 1 :: 'a::trans\text{-}reals)$
⟨*proof*⟩

lemmas *minus-one-not-infinity* [*simp*] = *infinity-noteq-minus-one* [*THEN not-sym*]

lemma *nullity-noteq-minus-one*[*simp*]: $\Phi \neq (- 1 :: 'a::trans\text{-}reals)$
⟨*proof*⟩

lemmas *minus-one-not-nullity* [*simp*] = *nullity-noteq-minus-one* [*THEN not-sym*]

lemma *zero-noteq-minus-infinity*[*simp*]: $0 \neq (- \infty :: 'a::trans\text{-}reals)$
⟨*proof*⟩

lemmas *minus-infinity-noteq-zero*[*simp*] = *zero-noteq-minus-infinity*[*THEN not-sym*]

lemma *uminus-zero*[*simp*]: $-(0\ ::'a::trans\text{-}reals) = 0$
⟨*proof*⟩

lemma *uminus-eq-zero-iff* [*simp*]: $!!\ x::'a::trans\text{-}reals.\ (-x = 0) = (x = 0)$
⟨*proof*⟩

lemma *minus-infinity-less-zero*[*simp*]: $(-\infty\ ::'a::trans\text{-}reals) < 0$
⟨*proof*⟩

lemma *minus-infinity-mult-infinity*[*simp*]: $(-\infty\ ::'a::trans\text{-}reals) * \infty = -\ \infty$
⟨*proof*⟩

lemma *not-zero-less-minus-infinity*[*simp*]: $\neg\ 0 <\ (-\infty\ ::'a::trans\text{-}reals)$
⟨*proof*⟩

lemma *one-noteq-minus-infinity*[*simp*]: $1 \neq (-\infty\ ::'a::trans\text{-}reals)$
⟨*proof*⟩

lemmas *minus-infinity-noteq-one*[*simp*] = *one-noteq-minus-infinity* [*THEN not-sym*]

lemma *infinity-noteq-minus-infinity*[*simp*]: $\infty \neq (-\infty\ ::'a::trans\text{-}reals)$
⟨*proof*⟩

lemmas *minus-infinity-noteq-infinity*[*simp*]
= *infinity-noteq-minus-infinity*[*THEN not-sym*]

lemma *nullity-noteq-minus-infinity*[*simp*]: $\Phi \neq (-\infty\ ::'a::trans\text{-}reals)$
⟨*proof*⟩

lemmas *minus-infinity-noteq-nullity*[*simp*]
= *nullity-noteq-minus-infinity*[*THEN not-sym*]

lemma *one-minus-one-eq-zero*[*simp*]: $(1::'a::trans\text{-}reals) - 1 = 0$
⟨*proof*⟩

lemma *less-irreflexive*[*simp*]: $\neg\ x < (x::'a::trans\text{-}reals)$
⟨*proof*⟩

TODO: less_transitive

lemma *not-less-zero-and-zero-less*:
$[\![\ x < (0\ ::'a::trans\text{-}reals);\ 0 < x\ ]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *less-zero-imp-not-zero*: $x < (0\ ::'a::trans\text{-}reals) \Longrightarrow x \neq 0$
⟨*proof*⟩

lemma *zero-less-imp-not-zero*: $(0 ::'a::trans\text{-}reals) < x \Longrightarrow x \neq 0$
⟨*proof*⟩

2.3 sign equivalences

lemma *sgn-negative-iff*:
!! $a ::'a::trans\text{-}reals.\ (sgn\ a = -\ 1) = (a < 0)$
⟨*proof*⟩

lemma *sgn-positive-iff*:
!! $a ::'a::trans\text{-}reals.\ (sgn\ a = 1) = (0 < a)$
⟨*proof*⟩

lemma *sgn-zero*[*simp*]: $sgn\ (0 ::'a::trans\text{-}reals) = 0$
⟨*proof*⟩

lemma *sgn-zero-iff*: $(sgn\ (x ::'a::trans\text{-}reals) = 0) = (x = 0)$
⟨*proof*⟩

lemma *sgn-nullity*[*simp*]: $sgn\ (\Phi ::'a::trans\text{-}reals) = \Phi$
⟨*proof*⟩

lemma *sgn-nullity-iff*: $(sgn\ (x ::'a::trans\text{-}reals) = \Phi) = (x = \Phi)$
⟨*proof*⟩

2.4 Algebraic stuff and inequalities...

lemma *infinity-add-infinity*[*simp*]: $(\infty ::'a::trans\text{-}reals) + \infty = \infty$
⟨*proof*⟩

lemma *minus-infinity-minus-infinity*[*simp*]: $-(\infty ::'a::trans\text{-}reals) - \infty = -\infty$
⟨*proof*⟩

lemma *minus-infinity-add-infinity*[*simp*]: $-(\infty ::'a::trans\text{-}reals) + \infty = \Phi$
⟨*proof*⟩

lemma *not-nullity-less*[*simp*]: $\neg\ \Phi\ < (x::'a::trans\text{-}reals)$
⟨*proof*⟩

lemma *not-less-nullity*[*simp*]: $\neg\ x\ < (\Phi::'a::trans\text{-}reals)$
⟨*proof*⟩

lemma *not-nullity-le-infinity*: $x \neq \Phi \Longrightarrow x < \infty \mid x = (\infty::'a::trans\text{-}reals)$
⟨*proof*⟩

lemma *less-infinity-iff*: $(x < \infty) = (x \neq (\infty::'a::trans\text{-}reals) \wedge x \neq \Phi)$
⟨*proof*⟩

lemma *not-nullity-minus-infinity-le*: $x \neq \Phi \Longrightarrow\ -\infty\ < x \mid x = (-\infty::'a::trans\text{-}reals)$
⟨*proof*⟩

lemma *minus-infinity-less-iff*: $(-\infty < x) = (x \neq (-\infty{::}'a{::}\mathit{trans\text{-}reals}) \wedge x \neq \Phi)$
⟨*proof*⟩

lemma *infinity-add-eq-nullity*:
$(\infty + x = \Phi) = (x = -\ (\infty\ {::}'a{::}\mathit{trans\text{-}reals}) \vee x = \Phi)$
⟨*proof*⟩

lemma *infinity-minus-eq-nullity*:
$(x - \infty = \Phi) = (x = \ (\infty\ {::}'a{::}\mathit{trans\text{-}reals}) \vee x = \Phi)$
⟨*proof*⟩

lemma *add-eq-nullity-iff*:
$(x + y = (\Phi{::}'a{::}\mathit{trans\text{-}reals})) =$
$(x = \Phi \vee y = \Phi \vee (x{=}\infty \wedge y = -\ \infty) \vee (x{=} -\ \infty \wedge y = \infty))$
⟨*proof*⟩

lemma *minus-nullity-eq-iff*:
$(x - y = (\Phi\ {::}'a{::}\mathit{trans\text{-}reals})) =$
$(x = \Phi \vee y = \Phi \vee (x{=}\infty \wedge y = \infty) \vee (x{=} -\ \infty \wedge y = -\infty))$

⟨*proof*⟩

lemma *add-infinity*:
$x + (\infty\ {::}'a{::}\mathit{trans\text{-}reals}) = \ (\mathit{if}\ x = \Phi \mid x = -\infty\ \ \mathit{then}\ \Phi\ \mathit{else}\ \infty)$
⟨*proof*⟩

lemma *add-infinity-not-eq*[*simp*]:
$(x\ {::}'a{::}\mathit{trans\text{-}reals}) + \infty \neq 0 \wedge (x\ {::}'a{::}\mathit{trans\text{-}reals}) + \infty \neq -\infty$
⟨*proof*⟩

lemma *subtract-infinity*:
$x - (\infty\ {::}'a{::}\mathit{trans\text{-}reals}) = (\mathit{if}\ x = \Phi \mid x = \infty\ \ \mathit{then}\ \Phi\ \mathit{else}\ -\infty)$
⟨*proof*⟩

lemma *subtract-infinity-not-eq*[*simp*]:
$(x\ {::}'a{::}\mathit{trans\text{-}reals}) - \infty \neq 0 \wedge (x\ {::}'a{::}\mathit{trans\text{-}reals}) - \infty \neq \infty$
⟨*proof*⟩

lemma *minus-nullD*: $x - y = (0\ {::}'a{::}\mathit{trans\text{-}reals}) \Longrightarrow x = y$
⟨*proof*⟩

lemma *add-cancel-right*: $[\![\ x + a = y + a;\ a \neq \Phi;\ a \neq \infty;\ a \neq -\infty\ ]\!]$
$\Longrightarrow x = (y\ {::}'a{::}\mathit{trans\text{-}reals})$
⟨*proof*⟩

lemma *add-cancel-left*: $[\![\ a + x = a + y;\ a \neq \Phi;\ a \neq \infty;\ a \neq -\infty\ ]\!]$
$\Longrightarrow x = (y\ {::}'a{::}\mathit{trans\text{-}reals})$
⟨*proof*⟩

lemma *add-eq-infinity-iff*:
$(x + y = (\infty::'a::trans\text{-}reals)) =$
$((x = \infty \ \wedge y \neq \ \Phi \wedge y \neq -\infty) \vee (y = \infty \ \wedge \ x \neq \ \Phi \wedge x \neq -\infty))$
⟨*proof*⟩

lemma *add-eq-minus-infinity-iff*:
$(x + y = (-\infty::'a::trans\text{-}reals)) =$
$((x = -\infty \ \wedge y \neq \ \Phi \wedge y \neq \infty) \vee (y = -\infty \ \wedge \ x \neq \ \Phi \wedge x \neq \infty))$
⟨*proof*⟩

lemma *reals-add*: !! $x::'a::trans\text{-}reals$.
$[\![\ x \neq \Phi; x \neq \infty; x \neq -\infty; \ \ y \neq \Phi; y \neq \infty; y \neq -\infty$
$]\!] \Longrightarrow (x + y) \neq \Phi \wedge (x + y) \neq \infty \ \wedge (x + y) \neq -\infty$
⟨*proof*⟩

2.5 Closure of reals under addition

lemma *reals-add-closed*: $[\![\ x \in reals; \ y \in reals \ ]\!] \Longrightarrow x + y \in reals$

⟨*proof*⟩

Distributivity of uminus over addition

lemma *uminus-distrib-add*: $-(x + y) = - \ x - (y \ ::'a::trans\text{-}reals)$
⟨*proof*⟩

lemma *uminus-distrib-minus*: $-(x - y) = - \ x + (y \ ::'a::trans\text{-}reals)$
⟨*proof*⟩

lemma *ordering-opp*: $(x - y < 0) = (x < (y::'a::trans\text{-}reals))$
⟨*proof*⟩

Trichotomy

lemma *trichotomy*:
$[\![\ x \neq \Phi; \ y \neq (\Phi::'a::trans\text{-}reals) \ ]\!] \Longrightarrow (x < y) \mid (x = y) \mid (y < x)$
⟨*proof*⟩

lemma *reals-uminus*: !! $x::'a::trans\text{-}reals$.
$[\![\ x \neq \Phi; x \neq \infty; x \neq -\infty \ ]\!] \Longrightarrow -x \ \neq \Phi \wedge -x \neq \infty \ \wedge -x \neq -\infty$
⟨*proof*⟩

2.6 Closure of reals under uminus

lemma *reals-uminus-closed*: $x \in reals \Longrightarrow -x \in reals$
⟨*proof*⟩

Closure of positives transreals under addition

lemma *zero-less-add*:
!! $a \ ::'a::trans\text{-}reals. \ [\![\ 0 < a; \ 0 < b \ ]\!] \Longrightarrow 0 < a + b$

$\langle proof \rangle$

lemma *reals-interval*: $reals = \{x.\ -\infty < x \wedge x < \infty\}$
$\langle proof \rangle$

lemma *reals-cases*: $x \in reals \Longrightarrow (x < 0) \mid x = 0 \mid (0 < x)$
$\langle proof \rangle$

lemma *not-infinity-less*[*simp*]: $\neg\ (\infty < (x::'a::trans\text{-}reals))$
$\langle proof \rangle$

lemma *not-less-minus-infinity*[*simp*]: $\neg\ ((x::'a::trans\text{-}reals) < -\infty)$
$\langle proof \rangle$

Left distributivy of multiplication over addition

lemma *distrib-left*:
$\neg\ ((c = \infty \vee c = -\ \infty) \wedge sgn\ a \neq sgn\ b \wedge (a + b \notin \{0,\Phi\}))$
$\Longrightarrow\ (a+b) * c = (a * c)+(b * (c::'a::trans\text{-}reals))$

$\langle proof \rangle$

lemma *zero-mult-not-less-zero*:$\neg\ ((0::'a::trans\text{-}reals) * x < 0)$
$\langle proof \rangle$

lemma *not-zero-less-zero-mult*:$\neg\ (0 < (0::'a::trans\text{-}reals) * x)$
$\langle proof \rangle$

lemma *zero-mult-eq-zero-or-nullity*: $(0::'a::trans\text{-}reals) * x = 0 \vee 0 * x = \Phi$
$\langle proof \rangle$

lemma *zero-mult-minus-one*: $(0::'a::trans\text{-}reals) * -\ 1 = 0$
$\langle proof \rangle$

lemma *zero-mult-zero*: $(0::'a::trans\text{-}reals) * 0 = 0$
$\langle proof \rangle$

Multiplicative inverse

lemma *mult-inverse*: $\llbracket\ a \neq 0;\ a:\ reals\ \rrbracket \Longrightarrow a * inverse\ a = (1::'a::trans\text{-}reals)$
$\langle proof \rangle$

lemma *zero-mult-real-not-zero*: $\llbracket\ (x::'a::trans\text{-}reals):\ reals;\ x \neq 0\ \rrbracket \Longrightarrow 0 * x = 0$
$\langle proof \rangle$

Multiplication with zero

lemma *zero-mult-reals*: $x : reals \Longrightarrow 0 * x = (0::'a::trans\text{-}reals) \wedge x * 0 = 0$
$\langle proof \rangle$

lemmas *zero-mult* = *zero-mult-reals* [*unfolded reals-def*, *simplified*]

lemma *mult-zero*: $x \neq \Phi \wedge x \neq \infty \wedge x \neq -\infty \Longrightarrow x * (0::'a::trans\text{-}reals) = 0$
⟨*proof*⟩

Multiplication with -1

lemma *minus-one-mult-uminus-reals*:
$x : reals \Longrightarrow (-\ 1\ ::'a::trans\text{-}reals) * x = -x$
⟨*proof*⟩

lemma *minus-one-mult-minus-one*[*simp*]: $(-\ 1\ ::'a::trans\text{-}reals) * -\ 1 = 1$
⟨*proof*⟩

lemma *minus-one-mult-uminus*:
$(-\ 1\ ::'a::trans\text{-}reals) * x = -x$
⟨*proof*⟩

lemma *minus-one-mult-minus-infinity*[*simp*]: $(-\ 1\ ::'a::trans\text{-}reals) * -\ \infty = \infty$
⟨*proof*⟩

lemma *mult-nullity-right*[*simp*]: $(x::'a::trans\text{-}reals) * \Phi = \Phi$
⟨*proof*⟩

lemma *zero-mult-minus-infinity*[*simp*]: $(0::'a::trans\text{-}reals) * -\ \infty = \Phi$
⟨*proof*⟩

lemma *minus-infinity-mult-zero*[*simp*]: $-\ \infty * (0::'a::trans\text{-}reals) = \Phi$
⟨*proof*⟩

lemma *uminus-mult-left*: $-((x::'a::trans\text{-}reals) * y) = (-\ x) * y$
⟨*proof*⟩

lemma *uminus-mult-right*: $-((x::'a::trans\text{-}reals) * y) = x * (-\ y)$
⟨*proof*⟩

lemma *uminus-mult-uminus* [*simp*]: $-(x::'a::trans\text{-}reals) * (-\ y) = x * y$
⟨*proof*⟩

lemma *zero-less-uminus-iff-less-zero*: $(0 < -\ x) = ((x::'a::trans\text{-}reals) < 0)$
⟨*proof*⟩

lemma *uminus-less-zero-iff-zero-less*: $(-x < 0) = (0 < (x::'a::trans\text{-}reals))$
⟨*proof*⟩

lemma *less-not-sym*: $(x::'a::trans\text{-}reals) < y \Longrightarrow \neg\ (y < x)$
⟨*proof*⟩

Closure of positive transreals under multiplication

lemma *mult-gt-zero-gt-zero*:
$!!\ a\ ::'a::trans\text{-}reals.\ [\![\ 0 < a;\ 0 < b\ ]\!] \Longrightarrow 0 < a * b$

$\langle proof \rangle$

Mixed-sign multiplication

lemma *mult-gt-zero-less-zero*:
$!!\ a::'a::trans\text{-}reals.\ [\![\ 0 < a;\ b < 0\ ]\!] \Longrightarrow a * b < 0$
$\langle proof \rangle$

lemma *mult-less-zero-gt-zero*:
$!!\ a::'a::trans\text{-}reals.\ [\![\ a < 0;\ 0 < b\ ]\!] \Longrightarrow a * b < 0$
$\langle proof \rangle$

Multiplication of two negative transreals

lemma *mult-less-zero-less-zero*:
$!!\ a::'a::trans\text{-}reals.\ [\![\ a < 0;\ b < 0\ ]\!] \Longrightarrow 0 < a * b$
$\langle proof \rangle$

lemma *infinity-mult*:
$\infty * (x::'a::trans\text{-}reals) =$
$(if\ x < 0\ then\ -\infty\ else\ if\ 0 < x\ then\ \infty\ else\ \Phi)$
$\langle proof \rangle$

lemma *mult-infinity*:
$(x::'a::trans\text{-}reals) * \infty =$
$(if\ x < 0\ then\ -\infty\ else\ if\ 0 < x\ then\ \infty\ else\ \Phi)$
$\langle proof \rangle$

lemma *minus-infinity-mult*:
$-\infty * (x::'a::trans\text{-}reals) =$
$(if\ x < 0\ then\ \infty\ else\ if\ 0 < x\ then\ -\infty\ else\ \Phi)$
$\langle proof \rangle$

lemma *mult-minus-infinity*:
$(x::'a::trans\text{-}reals) * -\infty=$
$(if\ x < 0\ then\ \infty\ else\ if\ 0 < x\ then\ -\infty\ else\ \Phi)$
$\langle proof \rangle$

Some inverse rules

lemma *inverse-infinity*[*simp*]: $inverse\ (\infty)\ = (0\ ::'a::trans\text{-}reals)$
$\langle proof \rangle$

lemma *inverse-nullity-iff*:
$!!\ x::'a::trans\text{-}reals.\ (inverse\ x = \Phi) = (x = \Phi)$
$\langle proof \rangle$

lemma *inverse-noteq-zero*:
$!!\ x::'a::trans\text{-}reals.\ [\![\ x \neq \infty;\ x \neq -\infty\ ]\!]\ \Longrightarrow inverse\ x \neq 0$
$\langle proof \rangle$

lemma *inverse-zero-iff*[*simp*]:

$!! \ x::'a::trans\text{-}reals. \ (inverse \ x = 0) = (x = \infty \mid x = -\infty)$
⟨proof⟩

lemma *inverse-infinity-iff* [*simp*]:
$!! \ x::'a::trans\text{-}reals. \ (inverse \ x = \infty) = (x = 0)$
⟨proof⟩

lemma *inverse-minus-infinity-iff* [*simp*]: $!! \ x::'a::trans\text{-}reals. \ (inverse \ x \neq -\infty)$
⟨proof⟩

2.7 Closure of positive reals under inverse

lemma *zero-less-inverse*:
$!! \ x \ ::'a::trans\text{-}reals. \ [\![\ 0 < x; \ x \neq \infty \ ]\!] \Longrightarrow 0 < inverse \ x$
⟨proof⟩

lemma *inverse-less-zero*:
$[\![\ (x::'a::trans\text{-}reals) < 0; \ x \neq -\infty \ ]\!] \Longrightarrow inverse \ x < 0$
⟨proof⟩

lemma *inverse-less-zero-iff* [*simp*]:
$(x::'a::trans\text{-}reals) \neq -\infty \Longrightarrow (inverse \ x < 0) = (x < 0)$
⟨proof⟩

lemma *zero-less-inverse-iff* [*simp*]:
$[\![\ (x::'a::trans\text{-}reals) \neq \infty; \ x \neq 0 \ ]\!] \Longrightarrow (0 < inverse \ x) = (0 < x)$
⟨proof⟩

lemma *less-not-nullity*:
$(x::'a::trans\text{-}reals) < y \Longrightarrow x \neq \Phi \wedge y \neq \Phi \wedge x \neq \infty \wedge y \neq -\infty$
⟨proof⟩

2.8 Closure of reals under multiplication

lemma *reals-mult-closed*: $[\![\ x : reals; \ y : reals \ ]\!] \Longrightarrow x * y : reals$
⟨proof⟩

end

3 A Model for Transarithmetic

theory *TransNumberModel*

imports *TransNumberAxclass Real*

begin

datatype $'a$ *trans-number* $= P\ 'a \mid$ *Infinity* $\mid$ *MinusInfinity* $\mid$ *Nullity*

lemma *inv-R-R*[*simp*]: $inv\ P\ (P\ x) = x$
⟨*proof*⟩

consts
primitive :: $'a$ *trans-number* $\Rightarrow$ *bool*
primrec
primitive $(P\ x) =$ *True*
primitive Infinity = *False*
primitive MinusInfinity = *False*
primitive Nullity = *False*

lemma *primitive-iff-exists*: *primitive* $x = (\exists\ r.\ x = P\ r)$
⟨*proof*⟩

lemma *primitiveD*: *primitive* $x \Longrightarrow \exists\ r.\ x = P\ r$
⟨*proof*⟩

instance *trans-number* :: (*zero*) *zero* ⟨*proof*⟩
instance *trans-number* ::(*type*) *infinity* ⟨*proof*⟩
instance *trans-number* ::(*type*) *nullity* ⟨*proof*⟩
instance *trans-number* :: (*plus*) *plus* ⟨*proof*⟩
instance *trans-number* :: (*minus*) *minus* ⟨*proof*⟩

defs (overloaded)
trans-number-zero-def: $0 == P\ 0$
trans-number-infinity-def: $\infty ==$ *Infinity*
trans-number-nullity-def: $\Phi ==$ *Nullity*

Note type annotations below, and overloaded symbol awkwardness

4 A model for axiomatic class trans_add

primrec
$P\ (x{::}'a{::}\{plus,minus\}) + y\ =$
(*if primitive* y *then* $P\ (x + inv\ P\ y)$ *else*
if $y = \infty$ *then* ∞ *else*

$\quad\quad$ *if* $y = -\infty$ *then* $-\infty$
$\quad\quad\quad\quad$ *else* Φ)

Infinity + (y::$'a$::{*plus*,*minus*} *trans-number*)
$\quad\quad$ = (*if primitive* $y \vee y = \infty$ *then* ∞ *else* Φ)

MinusInfinity +(y::$'a$::{*plus*,*minus*} *trans-number*)
$\quad\quad$ = (*if primitive* $y \vee y = -\infty$ *then* $-\infty$ *else* Φ)

Nullity-add-left:
Nullity +(y::$'a$::{*plus*,*minus*} *trans-number*) = Φ

primrec
uminus-P: $-$ P x = P ($-$ (x))
uminus-Infinity: $-$ *Infinity* = *MinusInfinity*
uminus-MinusInfinity: $-$ *MinusInfinity* = *Infinity*
uminus-Nullity: $-$ *Nullity* = *Nullity*

A6

defs (overloaded)
trans-number-subtract-def:
!! (y::$'a$::{*plus*,*minus*} *trans-number*).
$\quad$ x $-$ y == x + ($-$ y)

lemmas *trans-number-defs* =
trans-number-zero-def trans-number-infinity-def
trans-number-nullity-def
trans-number-subtract-def

lemma *trans-number-minus-infinity-sym-def*: *MinusInfinity* == $-\infty$
⟨*proof*⟩

lemmas *trans-number-sym-defs* =
trans-number-zero-def[*symmetric*]
trans-number-infinity-def [*symmetric*]
trans-number-minus-infinity-sym-def
trans-number-nullity-def [*symmetric*]
trans-number-subtract-def [*symmetric*]

lemma *primitive-zero*[*simp*]: *primitive 0*
⟨*proof*⟩

lemma *not-primitive-simps*[*simp*]:
$\neg$ *primitive* (∞) $\wedge$ $\neg$ *primitive* ($-\infty$) $\wedge$ $\neg$ *primitive* Φ
⟨*proof*⟩

lemma *primitive-iff*:
primitive x = ($x \neq \infty \wedge x \neq -\infty \wedge x \neq \Phi$)

⟨*proof*⟩

4.1 Distinctness of special values

lemma *P-neq-infinity*[*simp*]: $P\ x \neq \infty$
⟨*proof*⟩

lemma *P-neq-minus-infinity*[*simp*]: $P\ x \neq -\infty$
⟨*proof*⟩

lemma *P-neq-nullity*[*simp*]: $P\ x \neq \Phi$
⟨*proof*⟩

lemma *zero-neq-infinity*[*simp*]: $(0::'a::zero\ trans\text{-}number) \neq \infty$
⟨*proof*⟩

lemma *zero-neq-minus-infinity*[*simp*]:
$(0::'a::\{zero,minus\ \}\ trans\text{-}number) \neq -\infty$
⟨*proof*⟩

lemma *zero-neq-nullity*[*simp*]: $(0::'a::\{zero,minus\ \}\ trans\text{-}number) \neq \Phi$
⟨*proof*⟩

lemma *infinity-neq-minus-infinity*[*simp*]: $(\infty::'a::minus\ trans\text{-}number) \neq -\infty$
⟨*proof*⟩

lemma *infinity-neq-nullity*[*simp*]: $(\infty::'a::minus\ trans\text{-}number) \neq \Phi$
⟨*proof*⟩

lemma *minus-infinity-neq-nullity*[*simp*]: $(-\infty::'a::minus\ trans\text{-}number) \neq \Phi$
⟨*proof*⟩

declare *P-neq-infinity* [*THEN not-sym*,*simp*]
declare *P-neq-minus-infinity* [*THEN not-sym*,*simp*]
declare *P-neq-nullity* [*THEN not-sym*,*simp*]
declare *zero-neq-infinity* [*THEN not-sym*,*simp*]
declare *zero-neq-minus-infinity* [*THEN not-sym*,*simp*]
declare *zero-neq-nullity* [*THEN not-sym*,*simp*]
declare *infinity-neq-minus-infinity* [*THEN not-sym*,*simp*]
declare *infinity-neq-nullity* [*THEN not-sym*,*simp*]
declare *minus-infinity-neq-nullity* [*THEN not-sym*,*simp*]

lemma *P-add-P*[*simp*]: $P\ x + P\ y = P\ ((x\ ::'a::\{plus,minus\}) + y)$
⟨*proof*⟩

lemma *P-add-non-primitive*[*simp*]: $P\ x + \infty\ = \infty \wedge P\ x - \infty\ = -\infty \wedge\ P\ x + \Phi = \Phi$
⟨*proof*⟩

lemma *Infinity-add-left*[*simp*]:
$\infty + P\ x = (\infty::'a::\{plus,minus\}\ trans\text{-}number) \wedge$
$\infty + \infty = (\infty::'a\ trans\text{-}number) \wedge$

$\infty - \infty = (\Phi ::'a\ trans\text{-}number) \wedge$
$\infty + \Phi = (\Phi ::'a\ trans\text{-}number)$

$\langle proof \rangle$

lemma *MinusInfinity-add-left*[*simp*]:
$-\infty + P\ x = (-\infty::'a::\{plus,minus\}\ trans\text{-}number) \wedge$
$-\infty + \infty = (\Phi::'a\ trans\text{-}number) \wedge$
$-\infty - \infty = (-\infty\ ::'a\ trans\text{-}number) \wedge$
$-\infty + \Phi = (\Phi ::'a\ trans\text{-}number)$

$\langle proof \rangle$

lemma *uminus-simps*[*simp*]:
$-\ P\ x = (P\ (-\ x)\ ::'a::\{minus\}\ trans\text{-}number) \wedge$
$-\ (\ -\infty) = (\infty::'a\ trans\text{-}number) \wedge$
$-\ \Phi = (\Phi ::'a\ trans\text{-}number)$

$\langle proof \rangle$

A4

lemma *nullity-add-left*[*simp*]:
$\Phi + x = (\Phi ::'a::\{plus,minus\}\ trans\text{-}number)$

$\langle proof \rangle$

lemma *nullity-subtract-left*[*simp*]:
$\Phi - x = (\Phi ::'a::\{plus,minus\}\ trans\text{-}number)$

$\langle proof \rangle$

Axiom A5

lemma *addition-infinity-not-null*:
$[\![\ x \neq -\infty\ ;\ x \neq \Phi\ ]\!] \Longrightarrow (x::'a::\{plus,minus\}\ trans\text{-}number) + \infty = \infty$
$\langle proof \rangle$

A1

lemma *add-assoc*:
$((x::'a::ab\text{-}group\text{-}add\ trans\text{-}number) + y) + z = x + (y + z)$

$\langle proof \rangle$

A2

lemma *add-commute*:
$(x::'a::ab\text{-}group\text{-}add\ trans\text{-}number) + y = y + x$

$\langle proof \rangle$

lemma *add-left-commute*:
$a + (b + c) = b + (a + (c{::}'a{::}\textit{ab-group-add trans-number}))$
⟨*proof*⟩

theorems *trans-add-ac* = *add-assoc add-commute add-left-commute*

lemma *nullity-add-right*[*simp*]:
$x + \Phi = (\Phi\ {::}'a{::}\textit{ab-group-add trans-number})$
⟨*proof*⟩

A3

lemma *add-identity*[*simp*]:
$0 + (x{::}'a{::}\textit{ab-group-add trans-number}) = x$

⟨*proof*⟩

A7

lemma *bijectivity-of-uminus*[*simp*]:
$-(-(x{::}'a{::}\textit{ab-group-add trans-number})) = x$

⟨*proof*⟩

A8

lemma *additive-inverse*[*simp*]:
!! $(x{::}'a{::}\textit{ab-group-add trans-number})$.
primitive $x \Longrightarrow x - x = 0$

⟨*proof*⟩

A9

lemma *uminus-nullity*:
$-\ (\Phi\ {::}('a{::}\textit{minus})\ \textit{trans-number}) = \Phi$

⟨*proof*⟩

A10

lemma *subtraction-infinity-not-null*:
!! $(x{::}'a{::}\{\textit{plus},\textit{minus}\}\ \textit{trans-number})$.
$[\![\ x \neq \infty;\ x \neq \Phi\ ]\!] \Longrightarrow x - \infty = -\infty$

⟨*proof*⟩

instance *trans-number* :: (*ab-group-add*) *trans-add*
⟨*proof*⟩

5 Transnumber ordering

instance *trans-number* :: (*ord*) *ord* ⟨*proof*⟩

primrec
P-less: $P\ x < y = \ (primitive\ y \wedge (x{::}'a{::}\{minus,ord\}) < inv\ P\ y \mid y = \infty)$
Infinity-less: $(Infinity < (y{::}'a{::}\{minus,ord\}\ trans\text{-}number)) = False$
MinusInfinity-less:
$(MinusInfinity < y) = \ (y \neq \ -\infty \wedge y \neq (\Phi{::}'a{::}\{minus,ord\}\ trans\text{-}number))$
Nullity-less: $(Nullity < (y{::}'a{::}\{minus,ord\}\ trans\text{-}number)) = False$

defs (overloaded)
trans-number-le-def:
$(x{::}'a{::}\{minus,ord\}\ trans\text{-}number) <= y == (x < y \mid x = y)$

lemma *P-less-P*[*simp*]: $(P\ x < P\ y) = (x < (y{::}'a{::}\{minus,order\}))$
⟨*proof*⟩

lemma *P-less-non-primitive*[*simp*]: $(P\ x < \infty) \wedge \neg\ (P\ x < -\infty) \wedge \neg\ (P\ x < \Phi)$
⟨*proof*⟩

lemma *P-le*[*simp*]:
$(P\ x \leq P\ y = (x \leq (y{::}'a{::}\{order,minus\}))) \wedge$
$(P\ x \leq \infty) \wedge$
$\neg\ (P\ x \leq \ -\infty) \wedge$
$\neg\ (P\ x \leq \ \Phi)$
⟨*proof*⟩

lemma *not-infinity-less*[*simp*]:
$\neg\ ((\infty{::}'a{::}\{minus,ord\}\ trans\text{-}number) < x)$
⟨*proof*⟩

lemma *infinity-le* [*simp*]:
$((\infty{::}'a{::}\{minus,ord\}\ trans\text{-}number) \leq x) = (x = \infty)$
⟨*proof*⟩

lemma *le-infinity*[*simp*]: $x \leq (\infty{::}'a{::}\{minus,ord\}\ trans\text{-}number) = (x \neq \Phi)$
⟨*proof*⟩

lemma *minus-infinity-less*[*simp*]:
$(-(\infty{::}'a{::}\{minus,ord\}\ trans\text{-}number) < x) = (x \neq -\infty \wedge x \neq \Phi)$
⟨*proof*⟩

lemma *minus-infinity-le*[*simp*]:
$(-(\infty{::}'a{::}\{minus,ord\}\ trans\text{-}number) \leq x) = (x \neq \Phi)$
⟨*proof*⟩

lemma *not-nullity-less*[*simp*]: $\neg\ ((\Phi{::}'a{::}\{minus,ord\}\ trans\text{-}number) < x)$
⟨*proof*⟩

lemma *nullity-le*[*simp*]: $((\Phi{::}'a{::}\{minus,ord\}\ trans\text{-}number) \leq x) = (x = \Phi)$
⟨*proof*⟩

lemma *not-less-nullity*[*simp*]: $\neg$ (x < (Φ::$'a$::{*minus*,*ord*} *trans-number*))
⟨*proof*⟩

lemma *le-nullity*[*simp*]:
($x \leq$ (Φ::$'a$::{*minus*,*order*} *trans-number*)) = (x = Φ)
⟨*proof*⟩

lemma *zero-less-P*[*simp*]:
((*0*::$'a$::{*minus*,*ord*,*zero*} *trans-number*) < P x) = (*0* < x)
⟨*proof*⟩

A25

lemma *zero-less-infinity*[*simp*]: ((*0*::$'a$::{*minus*,*ord*,*zero*} *trans-number*) < ∞)
⟨*proof*⟩

lemma *zero-not-less-minus-infinity*[*simp*]:
$\neg$ ((*0*::$'a$::{*minus*,*ord*,*zero*} *trans-number*) < $-\infty$)
⟨*proof*⟩

lemma *irreflexive-less*[*simp*]: $\neg$ ((x::$'a$::{*minus*,*order*,*zero*} *trans-number*) < x)
⟨*proof*⟩

declare *P-less*[*simp del*]

A26

lemma *trans-number-ordering*:
($x - y$ > *0*) = (x > (y::$'a$::*lordered-ab-group trans-number*))
⟨*proof*⟩

A27 holds trivially since gt is simply a syntactic abbreviation

lemma *less-than-gt-than-eq*: (x < y) = (y > x)
⟨*proof*⟩

A28

lemma *quadrachotomoy*:
!!x::$'a$::{*ab-group-add*,*linorder*} *trans-number*.
xor [x < *0*, x = *0*, *0* < x, x= Φ]

⟨*proof*⟩

A29

lemma *pos-closure-add*:
!! x::$'a$::{*lordered-ab-group*} *trans-number*.
⟦ *0* < x ; *0* < y ⟧ $\Longrightarrow$ *0* < $x + y$

⟨*proof*⟩

lemma *trans-number-order-refl*:

$(x::'a::\{minus,order\}\ trans\text{-}number) \leq x$

$\langle proof \rangle$

lemma *trans-number-order-trans*:
$[\![(x::'a::\{minus,order\}\ trans\text{-}number) \leq y;\ y \leq z]\!] \Longrightarrow x \leq z$

$\langle proof \rangle$

lemma *trans-number-order-antisym*:
$[\![(x::'a::\{minus,order\}\ trans\text{-}number) \leq y;\ y \leq x]\!] \Longrightarrow x = y$

$\langle proof \rangle$

lemma *trans-number-less-le*:
$((x::'a::\{minus,order\}\ trans\text{-}number) < y) = (x \leq y \wedge x \neq y)$

$\langle proof \rangle$

axclass *minus-order* $\subseteq$ *order*, *minus*

instance *pordered-ab-group-add* $\subseteq$ *minus-order* $\langle proof \rangle$

instance *trans-number* :: (*minus-order*)*order*
$\langle proof \rangle$

lemma *ext-number-linear*:
$[\![(x ::'a::\{minus,linorder\}\ trans\text{-}number) \neq \Phi;\ y \neq \Phi]\!] \Longrightarrow x \leq y \mid y \leq x$
$\langle proof \rangle$

lemma *not-elem-conv*: $xs \subseteq \{x.\ x \neq a\} = (a \notin xs)$
$\langle proof \rangle$

5.1 Lattice-completeness of trans_numbers

NOTE: cannot express l.-c. of transnumbers as instance rule

lemma *ext-number-complete-lattice*:
lattice-complete $\{x ::\ real\ trans\text{-}number.\ x \neq \Phi\}$

$\langle proof \rangle$

6 A model for axiomatic class trans_mult

instance *trans-number* :: (*one*) *one* $\langle proof \rangle$
instance *trans-number* :: (*inverse*) *inverse* $\langle proof \rangle$
instance *trans-number* :: (*times*)*times* $\langle proof \rangle$

defs (overloaded)

trans-number-one-def: *1* == *P 1*

lemmas *trans-number-defs* =
trans-number-zero-def *trans-number-one-def*
trans-number-infinity-def *trans-number-nullity-def*
trans-number-subtract-def

lemmas *trans-number-sym-defs* =
trans-number-zero-def [*symmetric*]
trans-number-one-def [*symmetric*]
trans-number-infinity-def [*symmetric*]
trans-number-minus-infinity-sym-def
trans-number-nullity-def [*symmetric*]
trans-number-subtract-def [*symmetric*]

lemma *primitive-one*[*simp*]: *primitive 1*
⟨*proof*⟩

Warning: simpsets contain different mult laws with special cases for RHS

primrec
trans-mult-P:
$P\ (x{::}'a{::}\{zero,times,minus,ord\}) * y =$
$(\ \mathit{if\ primitive}\ y\ \mathit{then}\ P\ (x * \mathit{inv}\ P\ y)\ \mathit{else}$
$\mathit{if}\ (y = \infty \wedge x > 0) \mid (y = -\infty \wedge x < 0)\ \mathit{then}\ \infty\ \mathit{else}$
$\mathit{if}\ (y = \infty \wedge x < 0) \mid (y = -\infty \wedge x > 0)\ \mathit{then}\ -\infty$
$\mathit{else}\ \Phi)$

trans-mult-Infinity:
$\mathit{Infinity} * (y{::}'a{::}\{zero,times,minus,ord\}\ \mathit{trans\text{-}number}) =$
$(\mathit{if}\ (\mathit{primitive}\ y \wedge y > 0) \mid y = \infty\ \mathit{then}\ \infty\ \ \mathit{else}$
$\mathit{if}\ (\mathit{primitive}\ y \wedge y < 0) \mid y = -\infty\ \mathit{then}\ -\infty$
$\mathit{else}\ \Phi)$

trans-mult-MinusInfinity:
$\mathit{MinusInfinity} * (y{::}'a{::}\{zero,times,minus,ord\}\ \mathit{trans\text{-}number}) =$
$(\mathit{if}\ (\mathit{primitive}\ y \wedge y < 0) \mid y = -\infty\ \mathit{then}\ \infty\ \ \mathit{else}$
$\mathit{if}\ (\mathit{primitive}\ y \wedge y > 0) \mid y = \infty\ \mathit{then}\ -\infty$
$\mathit{else}\ \Phi)$

trans-mult-Nullity: $\mathit{Nullity} * (y{::}'a{::}\{zero,times,minus,ord\}\ \mathit{trans\text{-}number}) = \Phi$

lemma *P-mult-P*[*simp*]:
$P\ x * P\ y = P\ ((x\ {::}'a{::}\{zero,times,minus,linorder\}) * y)$

⟨*proof*⟩

A15 is first conjunct of mult_nullity, see also trans_mult_nullity

lemma *mult-nullity*[*simp*]:
$(\Phi\ {::}'a\ {::}\mathit{ordered\text{-}idom}\ \mathit{trans\text{-}number}) * x = \Phi \wedge x * \Phi = \Phi$

⟨*proof*⟩

A16 is fourth conjunct of mult_zero

lemma *mult-zero*[*simp*]:

$0 * P \ (x::'a::ordered\text{-}idom) = 0 \wedge$
$P\ x * 0 = 0 \wedge$
$(0 ::'a\ trans\text{-}number) * \infty = \Phi \wedge$
$\infty * (0 ::'a\ trans\text{-}number) = \Phi \wedge$
$(0 ::'a\ trans\text{-}number) * -\infty = \Phi \wedge$
$-\infty * (0 ::'a\ trans\text{-}number) = \Phi \wedge$
$\infty * \Phi = (\Phi ::'a\ trans\text{-}number)$

⟨*proof*⟩

lemma *mult-infinity*[*simp*]:

$\infty * (\infty::'a::ordered\text{-}idom\ trans\text{-}number) = \infty \wedge$
$(\infty::'a\ trans\text{-}number) * -\infty = -\infty \wedge$
$-\infty * (\infty::'a\ trans\text{-}number) = -\infty \wedge$
$-\infty * -\infty = (\infty::'a\ trans\text{-}number)$

⟨*proof*⟩

lemma *P-mult-infinity-less-zero*:

!! $x ::('a::ordered\text{-}idom).$
$x < 0 \Longrightarrow P\ x * \infty = -\infty \wedge \infty * P\ x = -\infty$

⟨*proof*⟩

lemma *P-mult-infinity-gt-zero*:

!! $x ::('a::ordered\text{-}idom).$
$0 < x \Longrightarrow P\ x * \infty = \infty \wedge \infty * P\ x = \infty$

⟨*proof*⟩

lemma *P-mult-MinusInfinity-less-zero*:

!! $x ::('a::ordered\text{-}idom).$
$x < 0 \Longrightarrow P\ x * -\infty = \infty \wedge -\infty * P\ x = \infty$

⟨*proof*⟩

lemma *P-mult-MinusInfinity-gt-zero*:

!! $x ::('a::ordered\text{-}idom).$
$0 < x \Longrightarrow P\ x * -\infty = -\infty \wedge -\infty * P\ x = -\infty$

⟨*proof*⟩

lemmas *P-mult-infinities* =
P-mult-infinity-less-zero P-mult-infinity-gt-zero

P-mult-MinusInfinity-less-zero P-mult-MinusInfinity-gt-zero

declare *trans-mult-P* [*simp del*]
trans-mult-Infinity [*simp del*]
trans-mult-MinusInfinity [*simp del*]
trans-mult-Nullity [*simp del*]

A13

lemma *mult-commute*:
$(x::'a::ordered\text{-}idom\ trans\text{-}number) * y = y * x$

⟨*proof*⟩

A12

lemma *mult-assoc*:
$((x::'a::ordered\text{-}idom\ trans\text{-}number) * y) * z = x * (y * z)$

⟨*proof*⟩

lemma *mult-left-commute*:
$x * (y * z) = y * (x * (z::'a::\ ordered\text{-}idom\ trans\text{-}number))$
⟨*proof*⟩

lemmas *trans-mult-ac* = *mult-assoc mult-commute mult-left-commute*

A14

lemma *mult-one-left*[*simp*]: $1 * x = (x::'a::\ ordered\text{-}idom\ trans\text{-}number)$

⟨*proof*⟩

lemma *mult-one-right*[*simp*]: $x * 1 = (x::'a::\ ordered\text{-}idom\ trans\text{-}number)$
⟨*proof*⟩

lemma *not-primitive-mult-infinity*[*simp*]:
$\neg\ (primitive\ (P\ (x::'a::ordered\text{-}idom) * \infty))$
⟨*proof*⟩

lemma *not-primitive-mult-MinusInfinity*[*simp*]:
$\neg\ (primitive\ (P\ (x::'a::ordered\text{-}idom) * -\infty))$
⟨*proof*⟩

6.1 Inverse and division

primrec
$inverse\ (P\ (x::'a::\{inverse,zero\})) = (if\ x = 0\ then\ \infty\ else\ P\ (inverse\ x))$
$inverse\ (Infinity::('a::\{inverse,zero\})\ trans\text{-}number) = 0$
$inverse\ (MinusInfinity::('a::\{inverse,zero\})\ trans\text{-}number) = 0$
$inverse\ (Nullity::('a::\{inverse,zero\})\ trans\text{-}number) = \Phi$

A17

defs (overloaded)
trans-number-divison-def:
$x\ /\ (y::'a::\{times,inverse\}\ trans\text{-}number) == x * inverse\ y$

lemma *inverse-Infinity*[*simp*]:
$inverse\ (\infty ::('a::\{inverse,zero\})\ trans\text{-}number) = 0$

⟨*proof*⟩

lemma *inverse-MinusInfinity*[*simp*]:
$inverse\ (-\infty ::('a::\{inverse,zero,minus\})\ trans\text{-}number) = 0$

⟨*proof*⟩

lemma *inverse-nullity*[*simp*]:
$inverse\ (\Phi ::('a::\{inverse,zero\})\ trans\text{-}number)= \Phi$

⟨*proof*⟩

A18

lemma *multiplicative-inverse*:
$[\![\ primitive\ (x::'a::ordered\text{-}field\ trans\text{-}number);\ x\ \neq\ 0\ ]\!] \Longrightarrow x\ /\ x = 1$

⟨*proof*⟩

A19

lemma *bij-inverse*:
$(x::'a::ordered\text{-}field\ trans\text{-}number) \neq -\infty \Longrightarrow inverse\ (inverse\ x) = x$

⟨*proof*⟩

A20

lemma *inverse-zero*[*simp*]:
$inverse\ (0::'a::ordered\text{-}field\ trans\text{-}number) = \infty$

⟨*proof*⟩

A21

lemma *inverse-MinusInfinity*[*simp*]:
$inverse\ \ (-\infty ::'a::ordered\text{-}field\ trans\text{-}number) = 0$

⟨*proof*⟩

A22

lemma *inverse-nullity*[*simp*]:
$inverse\ \ (\Phi::'a::ordered\text{-}field\ trans\text{-}number) = \Phi$

$\langle proof \rangle$

A23

lemma *positive-inf-mult*:
$(\infty * x = \infty) = (0 < (x::'a::ordered\text{-}field\ trans\text{-}number))$

$\langle proof \rangle$

A24

lemma *negative-inf-mult*:
$(\infty * x = -\infty) = (x < (0::'a::ordered\text{-}field\ trans\text{-}number))$

$\langle proof \rangle$

instance *trans-number* :: (*ordered-idom*) *sgn* $\langle proof \rangle$

defs (overloaded)
trans-number-sgn-def:
$sgn\ (a::'a::ordered\text{-}idom\ trans\text{-}number)$
$== (if\ 0 < a\ then\ 1\ else$
$if\ 0 = a\ then\ 0\ else$
$if\ a < 0\ then\ -\ 1\ else$
$(*\ a = \Phi\ *)\quad \Phi)$

instance *trans-number* :: (*ordered-idom*) *trans-sgn*
$\langle proof \rangle$

lemma *P-mult-infinity-neq-infinity-iff*:
$(P\ a * \infty \neq \infty) = (a \leq (0::'a::ordered\text{-}idom))$
$\langle proof \rangle$

lemma *P-mult-infinity-neq-MinusInfinity-iff*:
$(P\ a * \infty \neq -\infty) = (0 \leq (a::'a::ordered\text{-}idom))$
$\langle proof \rangle$

lemma *P-mult-MinusInfinity-neq-MinusInfinity-iff*:
$(P\ a * -\infty \neq -\infty) = (a \leq (0::'a::ordered\text{-}idom))$
$\langle proof \rangle$

lemma *P-mult-MinusInfinity-neq-infinity-iff*:
$(P\ a * -\infty \neq \infty) = (0 \leq (a::'a::ordered\text{-}idom))$
$\langle proof \rangle$

lemma *sgn-P*:
$sgn\ (P\ (x::'a::ordered\text{-}idom)) = (if\ x < 0\ then\ -\ 1\ else\ if\ x = 0\ then\ 0\ else\ 1)$
$\langle proof \rangle$

lemma *uminus-eq-iff*:
$(-x = (x::'a::ordered\text{-}idom)) = (x = 0)$

$\langle proof \rangle$

lemma *sgn-P-eq-iff*:
$!!\ (x{::}'a{::}ordered\text{-}idom)\ (y{::}'a).$
$(sgn\ (P\ x) = sgn\ (P\ y))$
$= (((x < 0) = (y < 0)) \wedge ((x = 0) = (y = 0)) \wedge$
$((0 < x) = (0 < (y{::}'a{::}ordered\text{-}idom))))$
$\langle proof \rangle$

lemma *sgn-zero*[*simp*]: $sgn\ (0{::}('a{::}ordered\text{-}idom\ trans\text{-}number)) = 0$
$\langle proof \rangle$

lemma *sgn-infinity*[*simp*]: $sgn\ (\infty\ {::}('a{::}ordered\text{-}idom\ trans\text{-}number)) = 1$
$\langle proof \rangle$

lemma *sgn-minus-infinity*[*simp*]: $sgn\ (-\infty\ {::}('a{::}ordered\text{-}idom\ trans\text{-}number)) =$
$-\ 1$
$\langle proof \rangle$

lemma *sgn-zero-iff*[*simp*]:
$(sgn\ (x{::}('a{::}ordered\text{-}idom\ trans\text{-}number)) = 0) = (x = 0)$
$\langle proof \rangle$

lemma *sgn-P-one-iff*[*simp*]:
$(sgn\ (P\ (x{::}'a{::}ordered\text{-}idom)) = 1) = (0 < x)$
$\langle proof \rangle$

lemma *sgn-P-minus-one-iff*[*simp*]:
$(sgn\ (P\ (x{::}'a{::}ordered\text{-}idom)) = -\ 1) = (x < 0)$
$\langle proof \rangle$

lemma *P-eq-zero*: $(P\ x = 0) = (x = 0)$
$\langle proof \rangle$

A29

lemma *distributivity*:
$!!\ a{::}('a{::}ordered\text{-}field\ trans\text{-}number).$
$\neg\ ((a = \infty \vee a = -\ \infty) \wedge sgn\ b \neq sgn\ c \wedge (b + c \notin \{0,\Phi\}))$
$\Longrightarrow a * (b{+}c) = (a * b){+}(a * c)$
$\langle proof \rangle$

instance *trans-number* :: (*ordered-field*) *trans-mult*
$\langle proof \rangle$

end