Transarithmetic

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Mobile Phones

During this scientific seminar, please leave your mobile telephone switched *on*.

- Use the calculator function on you mobile to divide any number of your choice by zero.
- What is the answer?

Every computer in the world, even those designed by IBM, makes some kind of error when dividing by zero.

- How cool would it be to be able to divide by zero?

The Nike Moment in Mathematics

Of course, we all know it is *impossible* to divide by zero.

- My teacher says it is impossible.
- My calculator cannot divide by zero.
- None of my friends can divide by zero.
- My pet lemming cannot divide by zero.

But the Greek goddess, Nike, says: “JUST DO IT.”

Fractions of Zero

For any number k write the fraction $\frac{k}{0}$

Did the sky fall in?

- No!

Cancellation

Cancel the largest, positive, common factor from the numerator and denominator.

$$\frac{k}{0} = \frac{\mathrm{sgn}(k) \times |k|}{0 \times |k|} = \frac{\mathrm{sgn}(k)}{0} \in \left\{ \frac{-1}{0}, \frac{0}{0}, \frac{1}{0} \right\}$$

Did the sky fall in?

- The Greatest Common Divisor algorithms cannot always cancel the Greatest Common Factor.

So what? Divisors and factors are different things!

Cancellation

- $$\frac{-7}{0} = \frac{\text{sgn}(-7) \times |-7|}{0 \times |-7|} = \frac{-1 \times 7}{0 \times 7} = \frac{-1}{0}$$
- $$\frac{0}{0} = \frac{\text{sgn}(0) \times |0|}{0 \times |0|} = \frac{0 \times 0}{0 \times 0} = \frac{0}{0}$$
- $$\frac{4}{0} = \frac{\text{sgn}(4) \times |4|}{0 \times |4|} = \frac{1 \times 4}{0 \times 4} = \frac{1}{0}$$

Cancellation

NOTE:

- $\operatorname{sgn}\left(\frac{0}{0}\right) = \frac{0}{0}$

- $\left|\frac{0}{0}\right| = \frac{0}{0}$

Naming

- I name this number, $1/0$, *infinity* and give it this sign ∞.
- I name this number, $-1/0$, *minus infinity* and give it this sign $-\infty$.
- I name this number, $0/0$, *nullity* and give it this sign Φ.

Did the sky fall in?

- No!

Standard Form

Given a fraction $f = n/d$, reduce it to standard form as follows.

- If $d < 0$ then form $f = \dfrac{-n}{-d}$
- Cancel the largest, positive, common factor from the numerator and denominator of f.

VERY IMPORTANT NOTE

- All arguments to the arithmetic operations must be reduced to standard form immediately *before* they are operated on.
- All results of the arithmetic operations must be reduced to standard form immediately *after* they have been produced.

Multiplication

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Did the sky fall in?

- No!

Multiplication

- $3 \times \infty = \frac{3}{1} \times \frac{1}{0} = \frac{3 \times 1}{1 \times 0} = \frac{3}{0} = \frac{1}{0} = \infty$
- $(-\pi) \times \infty = \frac{-\pi}{1} \times \frac{1}{0} = \frac{(-\pi) \times 1}{1 \times 0} = \frac{-\pi}{0} = \frac{-1}{0} = -\infty$
- $(-\infty) \times (-\infty) = \frac{-1}{0} \times \frac{-1}{0} = \frac{(-1) \times (-1)}{0 \times 0} = \frac{1}{0} = \infty$
- $0 \times \infty = \frac{0}{1} \times \frac{1}{0} = \frac{0 \times 1}{1 \times 0} = \frac{0}{0} = \Phi$
- $\Phi \times \infty = \frac{0}{0} \times \frac{1}{0} = \frac{0 \times 1}{0 \times 0} = \frac{0}{0} = \Phi$

Division

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

Did the sky fall in?

- No!

Division

- $\infty \div 3 = \frac{1}{0} \div \frac{3}{1} = \frac{1}{0} \times \frac{1}{3} = \frac{1 \times 1}{0 \times 3} = \frac{1}{0} = \infty$
- $3 \div \infty = \frac{3}{1} \div \frac{1}{0} = \frac{3}{1} \times \frac{0}{1} = \frac{3 \times 0}{1 \times 1} = \frac{0}{1} = 0$
- $\infty \div \infty = \frac{1}{0} \div \frac{1}{0} = \frac{1}{0} \times \frac{0}{1} = \frac{1 \times 0}{0 \times 1} = \frac{0}{0} = \Phi$
- $\infty \div 0 = \frac{1}{0} \div \frac{0}{1} = \frac{1}{0} \times \frac{1}{0} = \frac{1 \times 1}{0 \times 0} = \frac{1}{0} = \infty$
- $\Phi \div (-\pi) = \frac{0}{0} \div \frac{-\pi}{1} = \frac{0}{0} \times \frac{1}{-\pi} = \frac{0 \times 1}{0 \times (-\pi)} = \frac{0}{0} = \Phi$

Addition

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \text{ where } c = 0 \text{ and } a, b = \pm 1$$

$$\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (c \times b)}{b \times d} \text{ otherwise}$$

Did the sky fall in?

- No!

Addition

- $\infty + 3 = \frac{1}{0} + \frac{3}{1} = \frac{(1 \times 1) + (3 \times 0)}{0 \times 1} = \frac{1 + 0}{0} = \frac{1}{0} = \infty$
- $3 + \infty = \frac{3}{1} + \frac{1}{0} = \frac{(3 \times 0) + (1 \times 1)}{1 \times 0} = \frac{0 + 1}{0} = \frac{1}{0} = \infty$
- $\infty + \infty = \frac{1}{0} + \frac{1}{0} = \frac{1 + 1}{0} = \frac{2}{0} = \frac{1}{0} = \infty$
- $\infty + 0 = \frac{1}{0} + \frac{0}{1} = \frac{(1 \times 1) + (0 \times 0)}{0 \times 1} = \frac{1 + 0}{0} = \frac{1}{0} = \infty$
- $\Phi + \pi = \frac{0}{0} + \frac{\pi}{1} = \frac{(0 \times 1) + (\pi \times 0)}{0 \times 1} = \frac{0 + 0}{0} = \frac{0}{0} = \Phi$

Subtraction

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$$

Did the sky fall in?

- No!

Subtraction

- $\infty - 3 = \frac{1}{0} - \frac{3}{1} = \frac{1}{0} + \frac{-3}{1} = \frac{(1 \times 1) + ((-3) \times 0)}{0 \times 1}$

$= \frac{1 + 0}{0} = \frac{1}{0} = \infty$

- $3 - \infty = \frac{3}{1} - \frac{1}{0} = \frac{3}{1} + \frac{-1}{0} = \frac{(3 \times 0) + ((-1) \times 1)}{1 \times 0}$

$= \frac{0 - 1}{0} = \frac{-1}{0} = -\infty$

- $\infty - \infty = \frac{1}{0} - \frac{1}{0} = \frac{1}{0} + \frac{-1}{0} = \frac{1 - 1}{0} = \frac{0}{0} = \Phi$

Ordering

$$\frac{a}{b} > \frac{c}{d} \text{ if and only if } \frac{a}{b} - \frac{c}{d} > 0$$

Did the sky fall in?

- We have not said how infinity relates to zero.
- We now define $\infty > 0$
- We have not said how nullity relates to zero.
- We now define $\Phi \not< 0$, $\Phi \neq 0$, $\Phi \not> 0$
- This stops the sky falling in - even in calculus!

Ordering

Is $\infty > 16$?

$$\infty - 16 = \frac{1}{0} - \frac{16}{1} = \frac{1}{0} + \frac{-16}{1} = \frac{(1 \times 1) + ((-16) \times 0)}{0 \times 1}$$

$$= \frac{1 + 0}{0} = \frac{1}{0} = \infty > 0$$

Therefore $\infty > 16$.

Quiz

- How do you know that infinity is bigger than any real number?
- How do you know that real arithmetic is a special case of transarithmetic?
- How do you know that transarithmetic does not contradict anything in real arithmetic?
- How do you know that transarithmetic does not contradict anything in calculus?

Consistency

- Transarithmetic was axiomatised, the axioms were translated into predicates in higher order logic and a computer checked that all of the predicates are consistent.
- This gives a computer proof that transarithmetic is consistent.
- The computer was designed by IBM.

Conclusion

Of course, we all know it is *possible* to divide by zero.

- My teacher says it is possible.
- My new IBM calculator can divide by zero.
- All of my friends can divide by zero.
- My pet lemming can divide by zero.

And the Greek goddess, Nike, says: “JUST DO IT.”

- Now ring your Mum on your mobile and tell her you can divide by zero!