Topology of the Transreal Numbers

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Introduction

- Transreal arithmetic uses only the existing algorithms of arithmetic, but ignores the injunction not to divide by zero, in such a way that it preserves the maximum possible information about the magnitude and sign of numbers

- Transreal arithmetic has been proved consistent by translating its axioms into higher order logic and testing them in a computer proof system

- Over 40,000 people have obtained a copy of the published paper describing the consistency proof. No fault has been reported, but only one person has acknowledged trying to find a fault

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Agenda

- Transreal arithmetic
- Transreal topology
- Transreal calculus
- Conceivable physical consequences of transnumbers
- Computer exploitation of transnumbers
- Against NaN
- Work completed and underway
- Conclusion

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Transreal Numbers

Transreal numbers are *fractions,* f, of a real *numerator,* n, and a real *denominator,* d, such that $f = n/d$

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Strictly Transreal Numbers

The strictly transreal numbers are:

- Positive infinity, $\infty = \dfrac{1}{0}$
- Nullity, $\Phi = \dfrac{0}{0}$
- Negative infinity, $-\infty = \dfrac{-1}{0}$

Note that a fraction with a strictly transreal numerator and/or denominator simplifies to a fraction with a real numerator and denominator

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Canonical Form

The canonical form of a transreal number, n/d:

- Is $1/0$ when $n > 0$ and $d = 0$
- Is $0/0$ when $n = d = 0$
- Is $-1/0$ when $n < 0$ and $d = 0$
- Is n'/d' where $n = kn'$ and $d = kd'$ and $d' > 0$, where k is the highest, common, factor between n, d when n, d are both integral
- Is $(nd^{-1})/1$ when nd^{-1} is irrational

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Irrational Fractions

There are not enough names to name every real number so we often chose not to write irrational fractions in canonical form. For example:

- $f = \pi \div 2 = \dfrac{\pi}{1} \div \dfrac{2}{1} = \dfrac{\pi}{1} \times \dfrac{1}{2} = \dfrac{\pi \times 1}{1 \times 2} = \dfrac{\pi}{2}$

Here $\pi/2$ is not in canonical form. Nonetheless, we may write irrational fractions in canonical form by introducing an intermediate variable. For example:

- $f = \dfrac{n}{1}$ where $n = \dfrac{\pi}{2}$

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Division and Multiplication

Division is as easy as multiplication:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

- Division by zero occurs when at least one of b, c, d is zero

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Division and Multiplication

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

- I used to require that every number, a/b, c/d, d/c is reduced to canonical form before it is operated on, but it is possible to take a more relaxed approach:
- If the denominator of any argument to a multiplication is zero then as many factors $(-1)/(-1)$ are included as are needed to make all of the denominators non-negative

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Addition and Subtraction

Addition and subtraction are harder than division and multiplication:

- $\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (c \times b)}{b \times d}$ in general, but
- $\frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0}$ in particular
- Subtraction occurs when at least one of the arguments to addition is negative

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Addition and Subtraction

- $\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (c \times b)}{b \times d}$ in general, but
- $\frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0}$ in particular
- I used to require that every number a/b, c/d, $k/0$ is reduced to canonical form before it is operated on, but it is possible to take a more relaxed approach:
- If any argument to an addition has a zero denominator then that fraction is reduced to canonical form and as many factors $(-1)/(-1)$ are included as are needed to make all of the denominators non-negative

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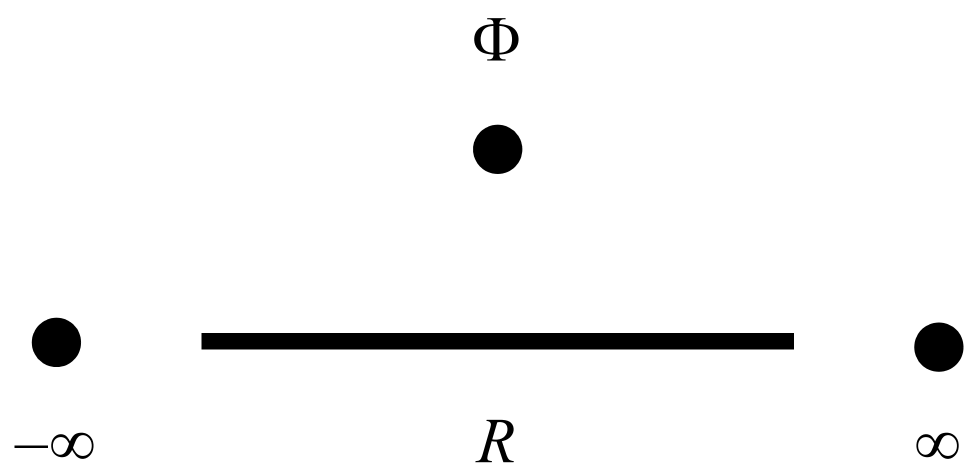
Topological Spaces

The open sets of the transreal numbers are generated from:

$$R,\ \{-\infty\},\ \{\infty\},\ \{\Phi\}$$

And can be visualised as:



- $\{-\infty\} \cup R \cup \{\infty\}$ is the extended-real line

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Continuity of Constant Functions

Any constant real function is continuous:

- $f(x) = k$ is continuous

But what of the constant functions:

- $f(x) = -\infty$, $f(x) = \infty$, $f(x) = \Phi$

Ordinary calculus cannot tell us anything about the continuity of $f(x) = \Phi$

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Continuity in Topological Spaces

We are interested in the continuity of constant, strictly transreal functions:

Let $S_1 = \langle P_1, T_1 \rangle$ be a topological space over the transreal numbers with $P_1 = R^T = R \cup \{-\infty, \infty, \Phi\}$ and T_1 being the set of subsets of P_1

Let $S_2 = \langle P_2, T_2 \rangle$ be the topological space with $P_2 = \{\Phi\}$ and $T_2 = \{\Phi\} \cup \{\varnothing\}$

Now, $f : P_1 \rightarrow P_2$ is the total, constant function $f(x) = \Phi$ for all transreal x in P_1

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Continuity in Topological Spaces

First, if $U = \{\Phi\}$ then $U \in T_2$ and $f^{-1}(U) = R^T \in T_1$

Second, the trivial case, if $U = \{\varnothing\}$ then $U \in T_2$ and $f^{-1}(U) = \varnothing \in T_1$

This completes the proof that f is continuous

Similarly, the functions $f(x) = -\infty$ and $f(x) = \infty$ are continuous on $R^T \to \{-\infty\}$ and $R^T \to \{\infty\}$, respectively

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Metric Spaces

Metric spaces are defined over a metric, m, which obeys four axioms:

$$m(a, b) = m(b, a) \qquad \text{[M1]}$$

$$m(a, b) \geq 0 \qquad \text{[M2]}$$

$$m(a, b) = 0 \Leftrightarrow a = b \qquad \text{[M3]}$$

$$m(a, b) + m(b, c) \geq m(a, c) \qquad \text{[M4]}$$

Replacing *greater-than-or-equals* with *not-less-than* generalises metric spaces to transmetric spaces

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Transmetric Spaces

Transmetric spaces are defined over a transmetric, t, which obeys four axioms:

$$t(a, b) = t(b, a) \qquad [T1]$$

$$t(a, b) \nless 0 \qquad [T2]$$

$$t(a, b) = 0 \Leftrightarrow a = b \qquad [T3]$$

$$t(a, b) + t(b, c) \nless t(a, c) \qquad [T4]$$

Transmetric spaces contain metric spaces as a subset so limiting processes continue to work for the transreal numbers

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The Euclidean Transmetric

The Euclidean transmetric, t, is:

$$t(a, b) = \begin{cases} 0 : a = b \\ \sqrt{(a-b)^2} : \text{otherwise} \end{cases}$$

Bar notation for the Euclidean transmetric:

$$|x, y| = t(x, y)$$

Bar notation for the Euclidean transmodulus:

$$|x| = t(x, 0)$$

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Calculus

- $\lim_{x \to a} f(x) = l$ if for every real $\varepsilon > 0$ there is some real $\delta > 0$ such that, for all real x, if $0 < |x, a| < \delta$, then $|f(x), l| < \varepsilon$
- $\lim_{x \to \infty} f(x) = l$ if for every real $\varepsilon > 0$ there is some real N such that, for all real $x > N$, it is the case that $|f(x), l| < \varepsilon$
- $\lim_{x \to \infty} f(x) = \infty$ if for every real $\varepsilon > 0$ there is some real N such that, for all real $x > N$, it is the case that $f(x) > \varepsilon$

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Calculus

A function can have a limit of Φ only in an interval where it is constant Φ because:

- The distance from Φ is zero or else nullity, but zero has a fixed value and nullity is incommensurate with any other number so the distance can never be reduced in any process, let alone a limiting process
- Growing unboundedly is not moving in the direction of Φ
- By contrast, a general function may have a limit of ∞ or else $-\infty$ because growing unboundedly can move monotonically in the direction of ∞ or else $-\infty$

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Calculus

In particular, the transmetric space has:

- $|-\infty, -\infty| = |\Phi, \Phi| = |\infty, \infty| = 0$

So calculus using the transmetric gives:

- $f(x) = -\infty$ is continuous
- $f(x) = \Phi$ is continuous
- $f(x) = \infty$ is continuous

Which is consistent with the topological and metric spaces and contains the whole of ordinary calculus

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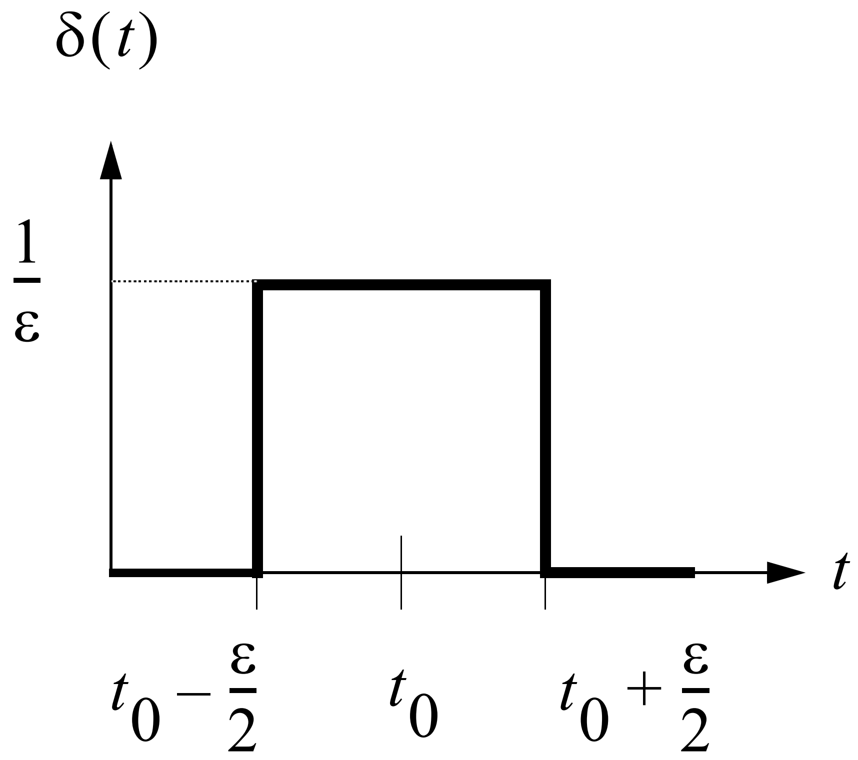
Dirac Delta

The Dirac Delta is the asymptote of the box function when epsilon tends to zero:



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Dirac Delta

- When epsilon tends *asyptotically* to zero, $\varepsilon \to 0$, the width tends *asymptotically* to zero, $w \to 0$, the height tends *asymptotically* to infinity, $h \to \infty$, and the area is everywhere *equal* to unity, $a = w \times h = \varepsilon/\varepsilon = 1$, because ε is everywhere a fixed real number greater than zero. Hence, the box function is the Dirac Delta

- When epsilon is *exactly* zero, $\varepsilon = 0$, the width is *exactly* zero, $w = \varepsilon = 0$, the height is *exactly* infinity $h = 1/\varepsilon = 1/0 = \infty$, and the area is *exactly* nullity, $a = w \times h = 0 \times \infty = (0/1) \times (1/0) = (0 \times 1)/(1 \times 0) = 0/0 = \Phi = 0/0 = \varepsilon/\varepsilon$, whence the box function *is not* the Dirac Delta

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Electron Self-Interaction

The interaction of a moving electron with the electric field passes through the Dirac Delta as a transfer function, but this gives the electron an infinite self-interaction

How can the infinity be removed from the physical equation?

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Electron Self-Interaction

- Use the box function in place of the Dirac Delta
- Observe that an electron has a small but non-zero radius
- Observe that a self-interaction is instantaneous
- Adopt an hypothesis linking transmathematics to physics:
- All nullity quantities lie outside our real-numbered part of the physical universe

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Electron Self-Interaction

- An electron e_i interacts with a different electron e_j in a small, but non-zero time, giving a box function area of unity so that a infinite real force is felt by the electron and the field
- An electron e_i interacts with itself in zero time, giving a box function area of nullity so that a nullity force is felt outside the extended-real universe and a zero force is felt inside the extended-real universe by the electron and the field. This removes the infinity from the entire universe

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Electron Self-Interaction

$$\varepsilon = 0 \qquad\qquad \varepsilon \to 0$$

$$\delta = \Phi \qquad\qquad \delta = 1$$

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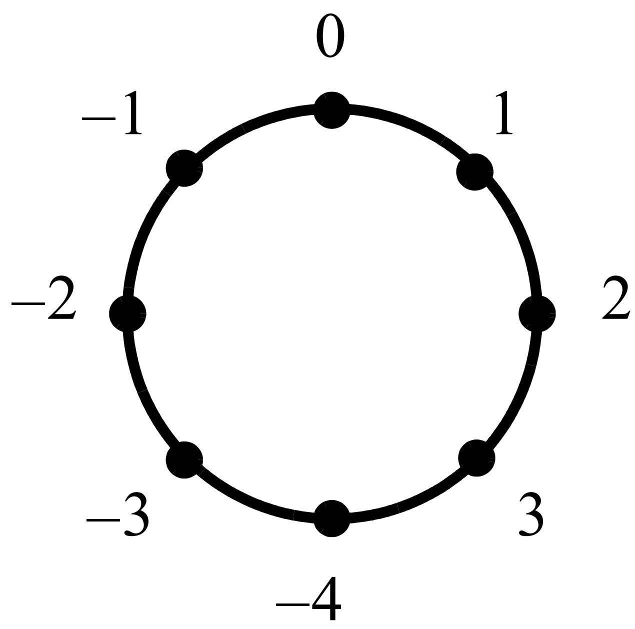
Two's Complement

Two's complement arithmetic is valid in itself, but using complement as negation is faulty in one case.



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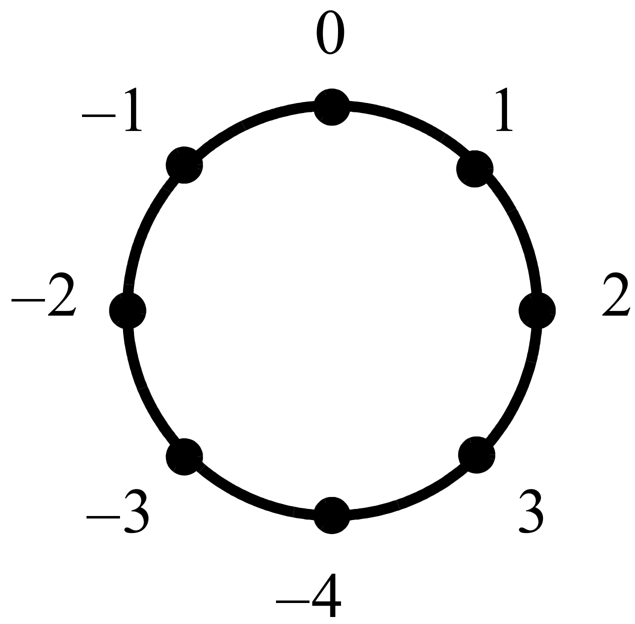
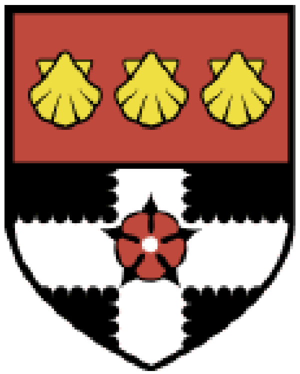
Two’s Complement



- The complement of the most negative number is not its negation –(–4) = –4
- Almost every computer suffers this weird-number fault



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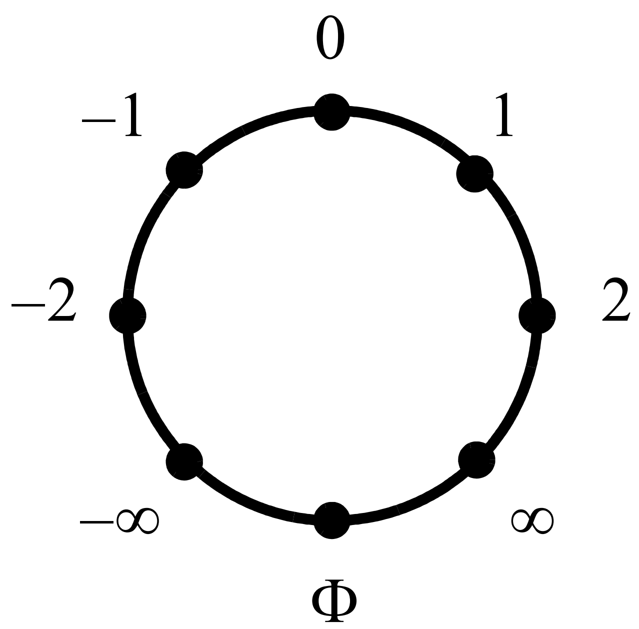
Trans Two's Complement



- The complement of the most negative number is now its negation $-(-\infty) = \infty$
- And the complement of nullity is its negation $-\Phi = \Phi$

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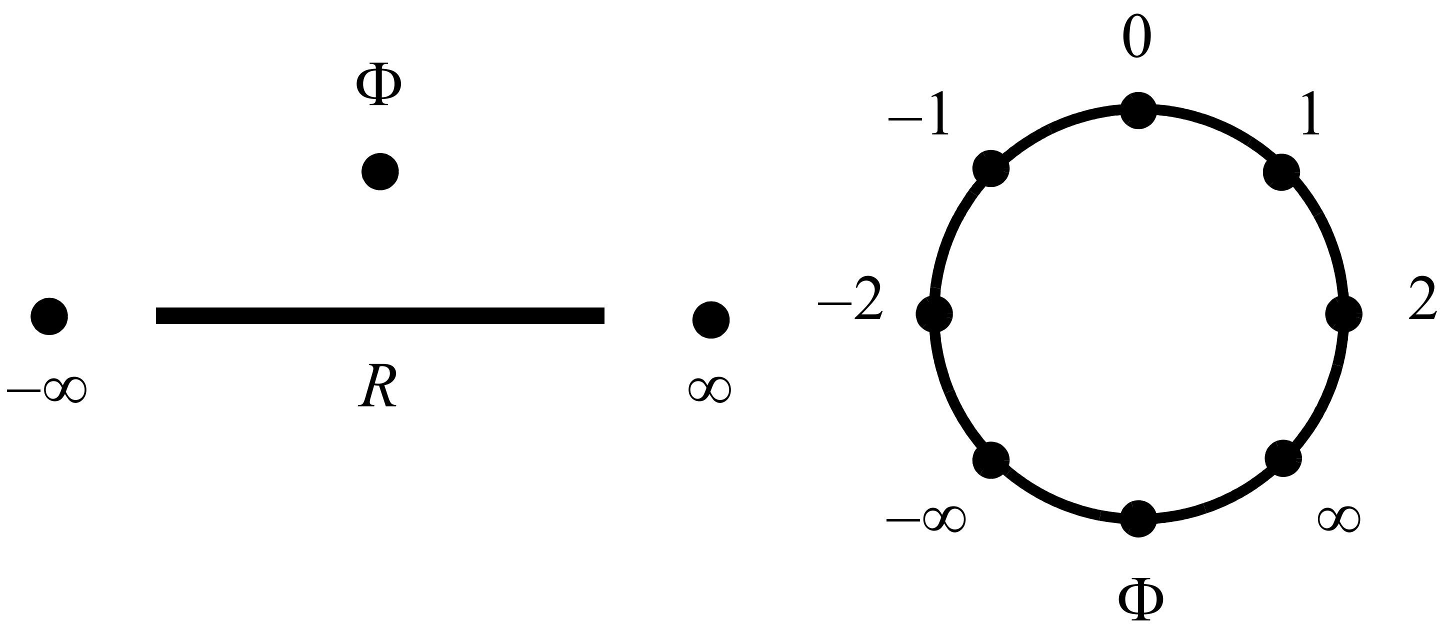
Trans Two's Complement

Trans two's complement removes the weird-number fault and preserves the topology of the transreal numbers



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Trans Two's Complement

Trans two's complement:

- Removes the two's complement fault
- Extends to multi-precision transintegers
- Extends to transfixed-point numbers
- Gives transfixed-point programming superior exception handling to floating-point arithmetic, reversing the current situation
- Extends to floating-point arithmetic so that it can match the exception handling of transfixed arithmetic

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Against NaN

Contemporary floating-point arithmetic uses NaN

- $\mathrm{NaN} \neq \mathrm{NaN}$ breaks the cultural stereotype amongst mathematicians, programmers, and the general public that any object is equal to itself. This makes NaN dangerous

- NaN breaks the Lambda calculus, because $\mathrm{NaN} \neq \mathrm{NaN}$ is incompatible with Lambda equality, rendering the theory of computation void, unless NaN is handled by adding unnecessary complexity to the calculus

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Against NaN

- There is no mathematical theory underlying NaN so every programmer is thrown back on his or her own resources. This encourages inconsistent uses of NaN in programming teams

By contrast:

- Nullity is equal to itself and has a consistent mathematical theory supporting it

- Therefore, nullity is much safer than NaN

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Software Engineering

- Software that performs all arithmetic in transreal numbers, or their generalisations, has no arithmetical exceptions
- Software that maps all language constructs, including memory management and peripheral handling, onto transreal numbers is total. That is, it has no exceptions
- Thus, transreal numbers make it easier to implement safety critical software

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Processor Design

A transreal processor:

- Has no exceptional states
- Has no error handling circuitry
- Never stalls on error
- Is smaller and/or faster than a conventional processor
- Can be proved correct by counting through its states in a small design
- Can be proved correct by algebraic induction on a practically sized design

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Work in Progress

Published:

- Transreal arithmetic
- Transreal trigonometry
- Transreal topology

Submitted:

- Transpower series

In preparation:

- Transreal differential calculus

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Conclusion

The transreal numbers are the best candidate for the principal augmentation of the real numbers because:

- They contain the real numbers and preserve the maximum possible information about the magnitude and sign of numbers on division by zero
- They appear to be consistent with all extensions of the real numbers
- They appear to support faster, cheaper, and safer computer processors than the real numbers or any extension of them
- They might solve some physical problems

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