Topology of the Transreal Numbers

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Introduction

- Transreal arithmetic uses only the existing algorithms of arithmetic, but ignores the injunction not to divide by zero, in such a way that it preserves the maximum possible information about the magnitude and sign of numbers

- Transreal arithmetic has been proved consistent by translating its axioms into higher order logic and testing them in a computer proof system

- Over 40,000 people have obtained a copy of the published paper describing the consistency proof. No fault has been reported, but only one person has acknowledged trying to find a fault
Agenda

• Transreal arithmetic
• Transreal topology
• Transreal calculus
• Conceivable physical consequences of transnumbers
• Computer exploitation of transnumbers
• Against NaN
• Work completed and underway
• Conclusion
Transreal Numbers

Transreal numbers are *fractions*, $f$, of a real *numerator*, $n$, and a real *denominator*, $d$, such that $f = n/d$
Strictly Transreal Numbers

The strictly transreal numbers are:

- Positive infinity, $\infty = \frac{1}{0}$
- Nullity, $\Phi = \frac{0}{0}$
- Negative infinity, $-\infty = \frac{-1}{0}$

Note that a fraction with a strictly transreal numerator and/or denominator simplifies to a fraction with a real numerator and denominator.
Canonical Form

The canonical form of a transreal number, \( n/d \):

- Is \( 1/0 \) when \( n > 0 \) and \( d = 0 \)
- Is \( 0/0 \) when \( n = d = 0 \)
- Is \( -1/0 \) when \( n < 0 \) and \( d = 0 \)
- Is \( n'/d' \) where \( n = kn' \) and \( d = kd' \) and \( d' > 0 \), where \( k \) is the highest, common, factor between \( n, d \) when \( n, d \) are both integral
- Is \( (nd^{-1})/1 \) when \( nd^{-1} \) is irrational
Irrational Fractions

There are not enough names to name every real number so we often chose not to write irrational fractions in canonical form. For example:

\[ f = \pi \div 2 = \frac{\pi}{1} \div \frac{2}{1} = \frac{\pi}{1} \times \frac{1}{2} = \frac{\pi \times 1}{1 \times 2} = \frac{\pi}{2} \]

Here \( \pi/2 \) is not in canonical form. Nonetheless, we may write irrational fractions in canonical form by introducing an intermediate variable. For example:

\[ f = \frac{n}{1} \text{ where } n = \frac{\pi}{2} \]
Division and Multiplication

Division is as easy as multiplication:

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}
\]

- Division by zero occurs when at least one of \( b, c, d \) is zero
Division and Multiplication

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}
\]

- I used to require that every number, \(a/b\), \(c/d\), \(d/c\) is reduced to canonical form before it is operated on, but it is possible to take a more relaxed approach:

- If the denominator of any argument to a multiplication is zero then as many factors \((-1)/(-1)\) are included as are needed to make all of the denominators non-negative
Addition and Subtraction

Addition and subtraction are harder than division and multiplication:

\[ \frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (c \times b)}{b \times d} \]  in general, but

\[ \frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{\pm 1 + \pm 1}{0} \]  in particular

• Subtraction occurs when at least one of the arguments to addition is negative
Addition and Subtraction

\[ \frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (c \times b)}{b \times d} \] in general, but

\[ \frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0} \] in particular

- I used to require that every number \( a/b, c/d, k/0 \) is reduced to canonical form before it is operated on, but it is possible to take a more relaxed approach:

- If any argument to an addition has a zero denominator then that fraction is reduced to canonical form and as many factors \((-1)/(-1)\) are included as are needed to make all of the denominators non-negative
Topological Spaces

The open sets of the transreal numbers are generated from:

$$R, \{-\infty\}, \{\infty\}, \{\Phi\}$$

And can be visualised as:

$$\{\infty\} \cup R \cup \{-\infty\}$$

is the extended-real line
Continuity of Constant Functions

Any constant real function is continuous:

- \( f(x) = k \) is continuous

But what of the constant functions:

- \( f(x) = -\infty, f(x) = \infty, f(x) = \Phi \)

Ordinary calculus cannot tell us anything about the continuity of \( f(x) = \Phi \)
Continuity in Topological Spaces

We are interested in the continuity of constant, strictly transreal functions:

Let $S_1 = \langle P_1, T_1 \rangle$ be a topological space over the transreal numbers with $P_1 = R^T = R \cup \{-\infty, \infty, \Phi\}$ and $T_1$ being the set of subsets of $P_1$

Let $S_2 = \langle P_2, T_2 \rangle$ be the topological space with $P_2 = \{\Phi\}$ and $T_2 = \{\Phi\} \cup \{\emptyset\}$

Now, $f : P_1 \rightarrow P_2$ is the total, constant function $f(x) = \Phi$ for all transreal $x$ in $P_1$
Continuity in Topological Spaces

First, if $U = \{ \emptyset \}$ then $U \in T_2$ and $f^{-1}(U) = R^T \in T_1$

Second, the trivial case, if $U = \{ \emptyset \}$ then $U \in T_2$ and $f^{-1}(U) = \emptyset \in T_1$

This completes the proof that $f$ is continuous

Similarly, the functions $f(x) = -\infty$ and $f(x) = \infty$ are continuous on $R^T \rightarrow \{-\infty\}$ and $R^T \rightarrow \{\infty\}$, respectively
Metric Spaces

Metric spaces are defined over a metric, \( m \), which obeys four axioms:

\[
m(a, b) = m(b, a) \tag{[M1]}
\]

\[
m(a, b) \geq 0 \tag{[M2]}
\]

\[
m(a, b) = 0 \iff a = b \tag{[M3]}
\]

\[
m(a, b) + m(b, c) \geq m(a, c) \tag{[M4]}
\]

Replacing greater-than-or-equals with not-less-than generalises metric spaces to transmetric spaces.
Transmetric Spaces

Transmetric spaces are defined over a transmetric, \( t \), which obeys four axioms:

\[
T1: \quad t(a, b) = t(b, a)
\]

\[
T2: \quad t(a, b) \neq 0
\]

\[
T3: \quad t(a, b) = 0 \iff a = b
\]

\[
T4: \quad t(a, b) + t(b, c) \neq t(a, c)
\]

Transmetric spaces contain metric spaces as a subset so limiting processes continue to work for the transreal numbers.
The Euclidean Transmetric

The Euclidean transmetric, $t$, is:

$$t(a, b) = \begin{cases} 
0 : a = b \\ 
\sqrt{(a - b)^2} : \text{otherwise}
\end{cases}$$

Bar notation for the Euclidean transmetric:

$$|x, y| = t(x, y)$$

Bar notation for the Euclidean transmodulus:

$$|x| = t(x, 0)$$
Calculus

- \( \lim_{x \to a} f(x) = l \) if for every real \( \varepsilon > 0 \) there is some real \( \delta > 0 \) such that, for all real \( x \), if \( 0 < |x, a| < \delta \), then \( |f(x), l| < \varepsilon \)

- \( \lim_{x \to \infty} f(x) = l \) if for every real \( \varepsilon > 0 \) there is some real \( N \) such that, for all real \( x > N \), it is the case that \( |f(x), l| < \varepsilon \)

- \( \lim_{x \to \infty} f(x) = \infty \) if for every real \( \varepsilon > 0 \) there is some real \( N \) such that, for all real \( x > N \), it is the case that \( f(x) > \varepsilon \)
Calculus

A function can have a limit of Φ only in an interval where it is constant Φ because:

• The distance from Φ is zero or else nullity, but zero has a fixed value and nullity is incommensurate with any other number so the distance can never be reduced in any process, let alone a limiting process

• Growing unboundedly is not moving in the direction of Φ

• By contrast, a general function may have a limit of ∞ or else −∞ because growing unboundedly can move monotonically in the direction of ∞ or else −∞
Calculus

In particular, the transmetric space has:

- \( |-\infty, -\infty| = |\Phi, \Phi| = |\infty, \infty| = 0 \)

So calculus using the transmetric gives:

- \( f(x) = -\infty \) is continuous
- \( f(x) = \Phi \) is continuous
- \( f(x) = \infty \) is continuous

Which is consistent with the topological and metric spaces and contains the whole of ordinary calculus.
Dirac Delta

The Dirac Delta is the asymptote of the box function when epsilon tends to zero:

$$\delta(t)$$

![Diagram showing the Dirac Delta function with parameters $t_0 - \frac{\varepsilon}{2}$, $t_0$, and $t_0 + \frac{\varepsilon}{2}$]
Dirac Delta

- When epsilon tends asymptotically to zero, $\varepsilon \to 0$, the width tends asymptotically to zero, $w \to 0$, the height tends asymptotically to infinity, $h \to \infty$, and the area is everywhere equal to unity, $a = w \times h = \varepsilon / \varepsilon = 1$, because $\varepsilon$ is everywhere a fixed real number greater than zero. Hence, the box function is the Dirac Delta.

- When epsilon is exactly zero, $\varepsilon = 0$, the width is exactly zero, $w = \varepsilon = 0$, the height is exactly infinity $h = 1 / \varepsilon = 1 / 0 = \infty$, and the area is exactly nullity, $a = w \times h = 0 \times \infty = (0 / 1) \times (1 / 0) = (0 \times 1) / (1 \times 0) = 0 / 0 = \Phi = 0 / 0 = \varepsilon / \varepsilon$, whence the box function is not the Dirac Delta.
Electron Self-Interaction

The interaction of a moving electron with the electric field passes through the Dirac Delta as a transfer function, but this gives the electron an infinite self-interaction.

How can the infinity be removed from the physical equation?
Electron Self-Interaction

- Use the box function in place of the Dirac Delta

- Observe that an electron has a small but non-zero radius

- Observe that a self-interaction is instantaneous

- Adopt an hypothesis linking transmathematics to physics:

- All nullity quantities lie outside our real-numbered part of the physical universe
Electron Self-Interaction

- An electron $e_i$ interacts with a different electron $e_j$ in a small, but non-zero time, giving a box function area of unity so that an infinite real force is felt by the electron and the field.

- An electron $e_i$ interacts with itself in zero time, giving a box function area of nullity so that a nullity force is felt outside the extended-real universe and a zero force is felt inside the extended-real universe by the electron and the field. This removes the infinity from the entire universe.
Electron Self-Interaction

\[ \varepsilon = 0 \quad \varepsilon \rightarrow 0 \]
\[ \delta = \Phi \quad \delta = 1 \]
Two’s Complement

Two’s complement arithmetic is valid in itself, but using complement as negation is faulty in one case.
Two’s Complement

- The complement of the most negative number is not its negation $-(-4) = -4$
- Almost every computer suffers this weird-number fault
Trans Two’s Complement

- The complement of the most negative number is now its negation $-(-\infty) = \infty$
- And the complement of nullity is its negation $-\Phi = \Phi$
Trans Two’s Complement

Trans two’s complement removes the weird-number fault and preserves the topology of the transreal numbers.
Trans Two’s Complement

Trans two’s complement:

- Removes the two’s complement fault
- Extends to multi-precision transintegers
- Extends to transfixed-point numbers
- Gives transfixed-point programming superior exception handling to floating-point arithmetic, reversing the current situation
- Extends to floating-point arithmetic so that it can match the exception handling of transfixed arithmetic
Against NaN

Contemporary floating-point arithmetic uses NaN

- NaN ≠ NaN breaks the cultural stereotype amongst mathematicians, programmers, and the general public that any object is equal to itself. This makes NaN dangerous

- NaN breaks the Lambda calculus, because NaN ≠ NaN is incompatible with Lambda equality, rendering the theory of computation void, unless NaN is handled by adding unnecessary complexity to the calculus
Against NaN

- There is no mathematical theory underlying NaN so every programmer is thrown back on his or her own resources. This encourages inconsistent uses of NaN in programming teams.

By contrast:

- Nullity is equal to itself and has a consistent mathematical theory supporting it.
- Therefore, nullity is much safer than NaN.
Software Engineering

- Software that performs all arithmetic in transreal numbers, or their generalisations, has no arithmetical exceptions

- Software that maps all language constructs, including memory management and peripheral handling, onto transreal numbers is total. That is, it has no exceptions

- Thus, transreal numbers make it easier to implement safety critical software
Processor Design

A transreal processor:

- Has no exceptional states
- Has no error handling circuitry
- Never stalls on error
- Is smaller and/or faster than a conventional processor
- Can be proved correct by counting through its states in a small design
- Can be proved correct by algebraic induction on a practically sized design
Work in Progress

Published:

- Transreal arithmetic
- Transreal trigonometry
- Transreal topology

Submitted:

- Transpower series

In preparation:

- Transreal differential calculus
Conclusion

The transreal numbers are the best candidate for the principal augmentation of the real numbers because:

- They contain the real numbers and preserve the maximum possible information about the magnitude and sign of numbers on division by zero
- They appear to be consistent with all extensions of the real numbers
- They appear to support faster, cheaper, and safer computer processors than the real numbers or any extension of them
- They might solve some physical problems