Dividing by Zero – How and Why?

Dr James A.D.W. Anderson
安德生

Agenda

- A word of comfort
- The story of complex numbers
- How to divide by zero
- Advantages for computing!
- Where do we go from here?

A Word of Comfort

- Dividing by zero is no more mysterious than finding the square root of a negative number
- Transreal arithmetic divides by zero using only accepted algorithms of arithmetic – so you already know how to divide by zero
- There is a machine proof that transreal arithmetic is consistent if real arithmetic is
- Every real result of mathematics stays the same, but there are some new, non-finite, results
- You can try out an implementation of transcomplex arithmetic

Can Calculators Divide by Zero?

- If you have an electronic calculator with you then turn it on and stand up
- Pick a number and divide it by zero on your calculator
- If your calculator shows an error or has crashed then sit down
- If your calculator is still working then multiply the current answer by zero
- If your calculator shows an error or has crashed then sit down
- Is there anyone left standing?

Can Computers Divide by Zero?

- My laptop can divide by zero. My FPGA model of a transreal computer can divide by zero. All of the PCs I have retrofitted with these arithmetics can divide by zero: trans-signed-integer, trans-two's-complement, transfloating-point, transreal, transcomplex
- Computers executing integer arithmetic cannot divide by zero
- Computers executing IEEE floating-point arithmetic cannot divide by zero. They produce a class of objects that are all Not a Number (NaN)

Can Computers Divide by Zero?

- The bridge of the missile cruiser, USS Yorktown, had networked computer control of navigation, engine monitoring, fuel control, machinery control, and damage control

Can Computers Divide by Zero?

- On September 21st, 1997, a sailor on the USS Yorktown entered a zero into a database field, causing a division by zero error which cascaded through the ship's network, crashing every computer on the network, and leaving the ship dead in the water for 2 hours 45 minutes
- The world would be a safer place if computers, calculators and people could divide numbers by zero, getting a number as an answer
- Coincidentally, I worked out how to do this in 1997

Complex Numbers

People used to believe that it is impossible to find the square root of a negative number

- $\sqrt{-4} = ?$
- $2 \times 2 = 4$
- $(-2) \times (-2) = 4$

Complex Numbers

- Invent a new number $i = j = \sqrt{-1}$
- Use only accepted algorithms of arithmetic
- BUT change the way the algorithms are applied by making addition non-absorptive, i.e. keep real and imaginary sums separate

Complex Numbers

- For example, complex multiplication is defined by:

$$\begin{aligned}(a+ib)(c+id) &= a(c+id)+ib(c+id)\\ &= ac+iad+ibc+i^2bd\\ &= ac+iad+ibc+(-1)bd\\ &= (ac-bd)+i(ad+bc)\\ &= k_1+ik_2\end{aligned}$$

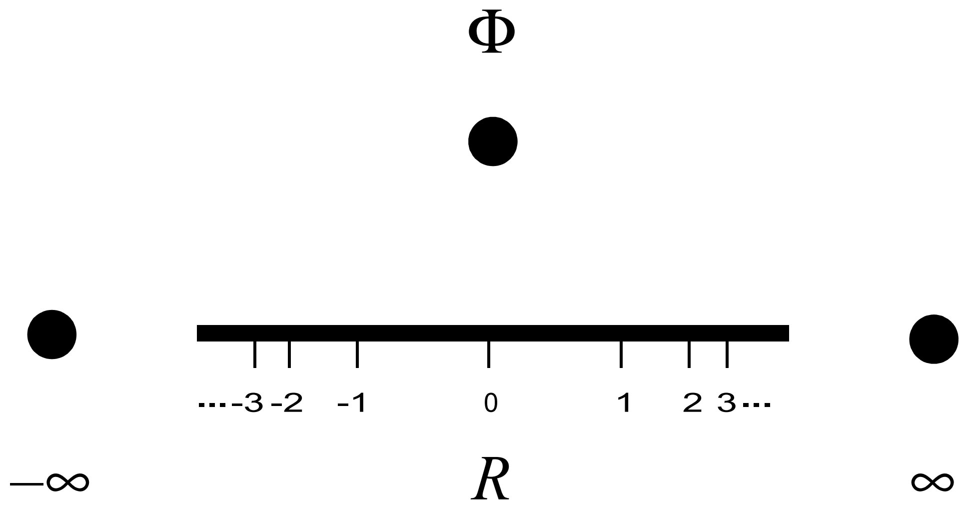
- Now $i2 \times i2 = i^2 4 = -1 \times 4 = -4$ so $\sqrt{-4} = i2$

Transreal Numbers

Invent some new numbers. For all $k > 0$ we define:

- $\infty = \dfrac{1}{0} \equiv \dfrac{k}{0}$
- $\Phi = \dfrac{0}{0}$
- $-\infty = \dfrac{-1}{0} \equiv \dfrac{-k}{0}$
- $0 = \dfrac{0}{1} \equiv \dfrac{0}{k} \equiv \dfrac{0}{-k}$

Transrealer Numbers



- Positive infinity, ∞, is the biggest transreal number
- Negative infinity, –∞, is the smallest transreal number
- Nullity, Φ, is the only transreal number that is not negative, not zero, and not positive

Transreal Numbers

- transreal
  - real
    - rational: $\frac{-1}{2}$ $\frac{0}{1}$ $\frac{2}{1}$
    - irrational: $\frac{-\sqrt{2}}{3}$ $\frac{2}{\pi}$
  - non-finite
    - infinite: $\frac{-1}{0}$ $\frac{1}{0}$
    - nullity: $\frac{0}{0}$

Transreal Fractions

A *transreal number* is a *transreal fraction* of the form $\frac{n}{d}$, where:

- n is the *numerator* of the fraction
- d is the *denominator* of the fraction
- n, d are *real numbers*
- $d \geq 0$
- Fractions with non-finite components simplify to the above form

Transreal Fractions

- An *improper transreal fraction,* $\frac{n}{-d}$, may have a negative denominator, $-d < 0$
- An improper transreal fraction is converted to a *proper transreal fraction* by multiplying both the numerator and denominator by minus one; or by negating both the numerator and the denominator, using subtraction; or it can be done, instrumentally, by moving the minus sign from the denominator to the numerator

$$\frac{n}{-d} = \frac{-1 \times n}{-1 \times (-d)} = \frac{-n}{-(-d)} = \frac{-n}{d}$$

Transreal Fractions

- Example: $\frac{2}{-3} = \frac{-1 \times 2}{-1 \times (-3)} = \frac{-2}{3}$
- Example: $\frac{0}{-1} = \frac{-0}{-(-1)} = \frac{0}{1}$
- Example: $\frac{x}{-y} = \begin{cases} \frac{x}{y} & : \; y = 0 \\ \frac{-x}{y} & : \; \text{otherwise} \end{cases}$

Transreal Multiplication

Two *proper transreal fractions* are multiplied like this:

- $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$
- Example: $3 \times \infty = \frac{3}{1} \times \frac{1}{0} = \frac{3 \times 1}{1 \times 0} = \frac{3}{0} = \infty$
- Example: $0 \times \infty = \frac{0}{1} \times \frac{1}{0} = \frac{0 \times 1}{1 \times 0} = \frac{0}{0} = \Phi$
- Example: $\frac{1}{2} \times \frac{3}{5} = \frac{1 \times 3}{2 \times 5} = \frac{3}{10}$

Transreal Division

Two *proper transreal fractions* are divided like this:

- $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$
- Example: $\infty \div 3 = \frac{1}{0} \div \frac{3}{1} = \frac{1}{0} \times \frac{1}{3} = \frac{1 \times 1}{0 \times 3} = \frac{1}{0} = \infty$
- Example:

$$\infty \div (-3) = \frac{1}{0} \div \frac{-3}{1} = \frac{1}{0} \times \frac{1}{-3} = \frac{1}{0} \times \frac{-1 \times 1}{-1 \times (-3)}$$

$$= \frac{1}{0} \times \frac{-1}{3} = \frac{1 \times (-1)}{0 \times 3} = \frac{-1}{0} = -\infty$$

Transreal Division

- Example: $\frac{1}{2} \div \frac{5}{3} = \frac{1}{2} \times \frac{3}{5} = \frac{1 \times 3}{2 \times 5} = \frac{3}{10}$

Transreal Addition

Two *proper transreal fractions* are added like this:

- $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$, except that:
- $(\pm\infty) + (\pm\infty) = \dfrac{\pm 1}{0} + \dfrac{\pm 1}{0} = \dfrac{(\pm 1) + (\pm 1)}{0}$

Transreal Addition

- $(\pm\infty) + (\pm\infty) = \frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0}$

Examples:

- $\infty + \infty = \frac{1}{0} + \frac{1}{0} = \frac{1 + 1}{0} = \frac{2}{0} = \infty$
- $(-\infty) + (-\infty) = \frac{-1}{0} + \frac{-1}{0} = \frac{(-1) + (-1)}{0} = \frac{-2}{0} = -\infty$
- $\infty + (-\infty) = \frac{1}{0} + \frac{-1}{0} = \frac{1 + (-1)}{0} = \frac{0}{0} = \Phi$

Transreal Addition

- $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

Examples:

- $\frac{2}{3} + \infty = \frac{2}{3} + \frac{1}{0} = \frac{2 \times 0 + 3 \times 1}{3 \times 0} = \frac{3}{0} = \infty$
- $\frac{2}{3} + \Phi = \frac{2}{3} + \frac{0}{0} = \frac{2 \times 0 + 3 \times 0}{3 \times 0} = \frac{0}{0} = \Phi$
- $\frac{2}{3} + \frac{4}{5} = \frac{2 \times 5 + 3 \times 4}{3 \times 5} = \frac{22}{15}$

Transreal Subtraction

Two *proper transreal fractions* are subtracted like this:

- $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{a}{b} + \dfrac{-c}{d}$

Examples:

- $\infty - \infty = \dfrac{1}{0} - \dfrac{1}{0} = \dfrac{1}{0} + \dfrac{-1}{0} = \dfrac{1 + (-1)}{0} = \dfrac{1 - 1}{0} = \dfrac{0}{0} = \Phi$
- $\dfrac{1}{2} - \dfrac{3}{5} = \dfrac{1}{2} + \dfrac{-3}{5} = \dfrac{(1 \times 5) + (2 \times (-3))}{2 \times 5} = \dfrac{5 + (-6)}{10} = \dfrac{-1}{10}$

Transreal Arithmetic

- Transreal arithmetic is a superset of real arithmetic
- Transreal arithmetic is *total* – every operation of transreal arithmetic can be applied to any transreal numbers with the result being a transreal number
- Real arithmetic is *partial* – it fails on division by zero and on each of the infinitely many mathematical consequences of division by zero

Transreal Associativity

Transreal arithmetic is totally associative over addition and multiplication:

- $a + (b + c) = (a + b) + c$
- $a \times (b \times c) = (a \times b) \times c$

Transreal Commutativity

Transreal arithmetic is totally commutative over addition and multiplication:

- $a + b = b + a$
- $a \times b = b \times a$

Transreal Distributivity

Transreal arithmetic is only partially distributive:

$$a \times (b + c) = (a \times b) + (a \times c)$$

- If a is finite or nullity then a distributes over any $b + c$
- If a is infinity or minus infinity then a distributes if $b + c = \Phi$ or $b + c = 0$ or b and c have the same sign
- Two numbers have the same sign if they are both positive, both negative, both zero, or both nullity

Transreal Distributivity

Despite the fact that transreal arithmetic is only partially distributive, it is still a total arithmetic because we can always evaluate any arithmetical expressions, including both of:

- $a \times (b + c)$
- $(a \times b) + (a \times c)$

It's just that these two expressions might, or might not, be equal!

- Computational paths generally bifurcate into a distributive and a non-distributive branch

Moral

- The arithmetic you have just seen has been taught to 12 year old children in England
- These children understand infinity and nullity
- These children use an arithmetic that never fails
- What do you want for your children?
- What do you want for your computers?
- What do you want for your self?

Advantages for Computing!

- All mathematical software can be extended to use transreal or transcomplex numbers

Advantages for Computing!

Every syntactically correct transarithmetical expression is semantically correct so:

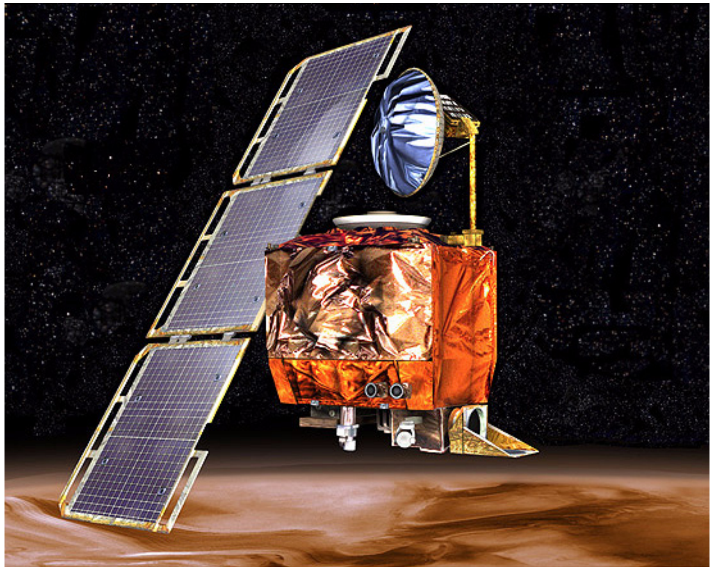
- Compilers can perform full type checking
- There are no arithmetical run-time errors
- Any Turing program can be executed without any logical run-time errors
- Pipelines are Turing computable so, in a finite machine of sufficient size, pipelines never break and entire programs can be pipelined

Advantages for Computing!

- Transreal arithmetic removes an intrinsic bug from two's complement arithmetic, making both hardware and software safer
- Floating-point hardware can have all wasted states re-allocated to transreal numbers, thereby improving arithmetical range or precision
- Floating-point hardware and software can have simplified ordering operations and exceptions
- It is possible to remove all exceptions from floating-point hardware by reserving an inexact flag in the number representation and by knowing the rounding mode

Mars

- NASA’s Climate Orbiter, which cost $125 M, crashed into the surface of Mars on 23 September, 1999, because a computer program mixed up foot-pound-second units with metre-kilogram-second units



- “People sometimes make errors,” Edward Weiler, NASA's Associate Administrator for Space Science

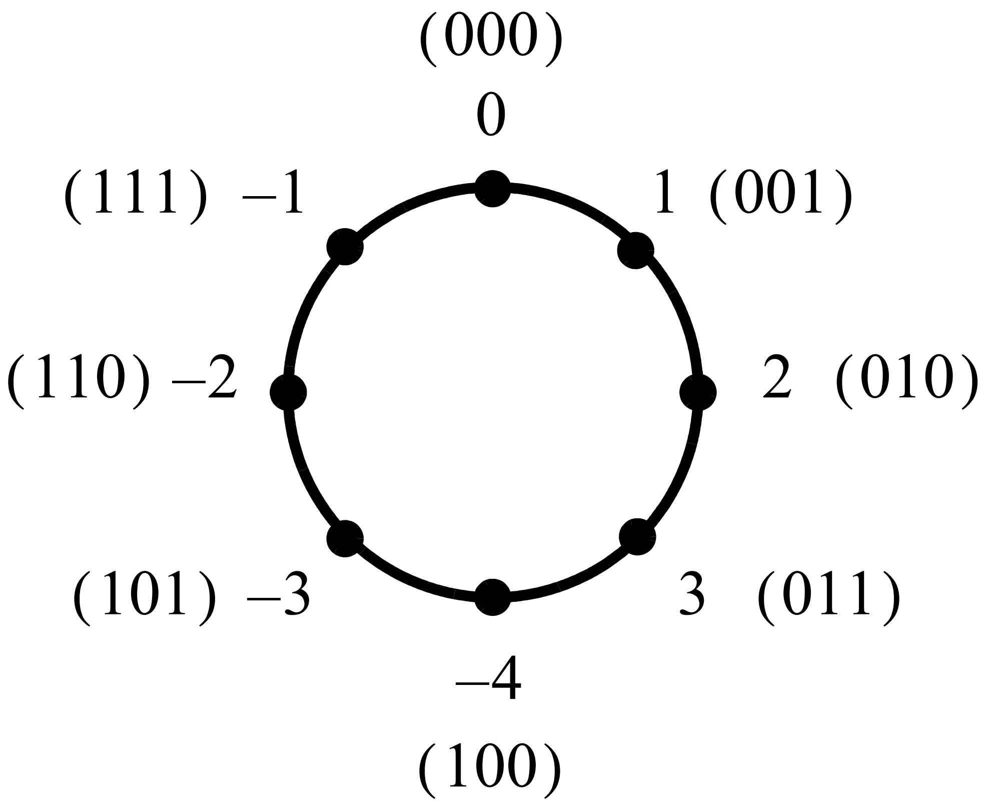
Mars

- This bug could have been caught if the compiler had used dimensional analysis
- All ordinary type checking and ordinary dimensional analysis fail on division by zero
- But all syntactically correct sentences of transarithmetic are semantically correct – which means that a compiler can always check every possible evaluation of the transarithmetic in any program
- How much would NASA pay for a compiler that can always apply dimensional analysis?

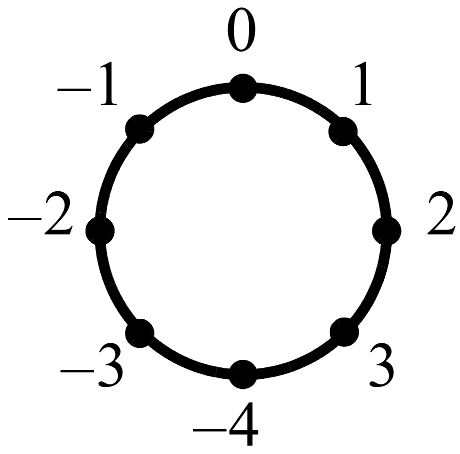
Mars

- Would NASA pay more for a compiler that implements physical units so that arithmetic is more accurate?

Two's Complement

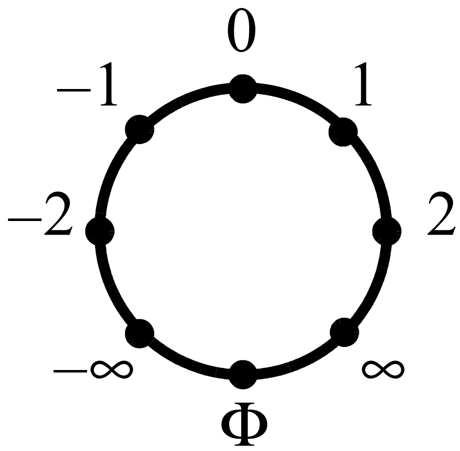


Two's Complement



- The complement of the most negative number is not its negation –(–4) = –4
- Almost every computer suffers this weird-number fault

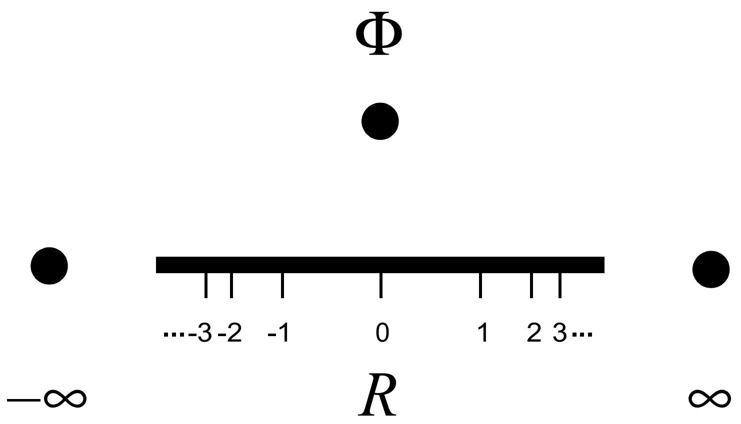
Trans Two’s Complement

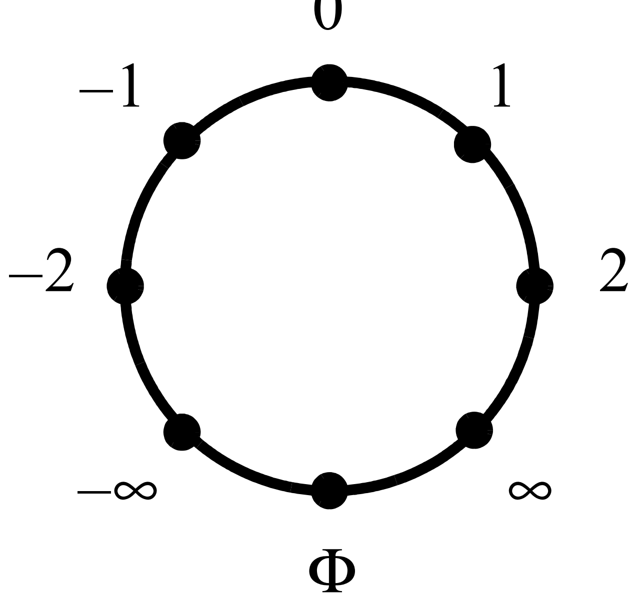


- The complement of the most negative number is now its negation $-(-\infty) = \infty$
- And the complement of nullity is its negation $-\Phi = \Phi$

Trans Two’s Complement

- Trans two’s complement removes the weird-number fault and preserves the topology of the transreal numbers





Trans Two’s Complement

- Trans two’s complement gives transinteger and transfixed-point programming superior exception handling to ordinary floating-point arithmetic

- The topology of transreal numbers extends to floating-point arithmetic so that it can match, and exceed, the exception handling of transinteger and transfixed arithmetic

Apocryphal Story

Apocryphal Story

- Once upon a time a Polaris missile was test fired. On launch it experienced severe turbulence and the guidance system instructed a maximal correction in the opposite direction to the turbulence

- The maximal correction happened to be the weird number. It was multiplied by -1, to be a correction in the opposite direction, but it stayed in the same direction, because it was the weird number

- Turbulence continued and the guidance system regained control of the missile’s attitude and sent it on its ballistic course, on the *opposite* bearing from the one intended

Apocryphal Story

- How much would you pay to have strategic nuclear missiles fly in the right direction?

Floating-Point

- IEEE 754 defines floating point numbers in terms of three of bit fields that encode the Sign (S), Exponent (E) and Mantissa (M)

- In general, a floating point number $n = -1^S 2^E M$, but bit patterns are reserved for –0, –∞, ∞, NaN_i where $i = 2^{m+1} - 2$ with m being the number of bits explicitly represented in the mantissa. The "+1" arises from the sign bit and the "–2" from –∞ and +∞

- IEEE arithmetic encodes -0, but -0 does not occur in transreal arithmetic so transfloating-point arithmetic reallocates the binary code for -0 to Φ

Floating-Point

- Nullity now lies in the middle of the lexical collation range of floating-point numbers so sorting routines must handle the unique nullity as a special case (IEEE sorting routines must handle all NaNs as special cases)
- Notice that IEEE arithmetic collates numbers in reverse order because S is the most significant bit and $S = 1$ encodes negative numbers
- IEEE arithmetic uses reverse collation so that the binary codes for integer zero and floating-point positive-zero are identical. As many computers have an integer test zero instruction this makes positive-zero comparisons quick

Floating-Point

- Transfloating-point arithmetic uses the most positive binary code for ∞ and the most negative binary code for –∞
- IEEE arithmetic has $2^{m+1} - 2$ NaNs, but transreal arithmetic has no NaNs so all of these states can be reallocated to real numbers
- Taking the reallocated codes as signed numbers means that there are $2^m - 1$ new mantissas. This is almost one binade: $2^m - 1$ bit patterns in a near binade as against 2^m bit patterns in an entire binade

Floating-Point

- Leaving the exponent bias unchanged takes this near binade with a positive exponent so as to almost double the arithmetical range of the real floating-point numbers
- Incrementing the bias takes this near binade with a positive exponent, but frees up an entire binade with a negative exponent, thereby increasing precision by delaying underflow to denormal numbers
- Only one of these options can be taken in one thread: either increase the range or else increase the precision

Floating-Point

Name	Common Name	m	Wasted NaN States
binary16	half precision	10	2 046
binary32	single precision	23	16 777 214
binary64	double precision	52	9 007 199 254 740 990
binary128	quadruple precision	112	10 384 593 717 069 655 257 060 992 658 440 190

Floating-Point

- The IEEE standard defines four relational operations: *less-than* (<), *equal* (=), *greater-than* (>), *unordered* (?)
- Transreal arithmetic defines three relational operations: *less-than* (<), *equal* (=), *greater-than* (>)

Floating-Point

- The IEEE standard defines 14 composite relations:
  =, ?<>, >, >=, <, <=, ?, <>, <=>, ?>, ?>=, ?<, ?<=, ?=

- The IEEE standard defines negations of 12 out of 14 of the composite relations:
  NOT(>), NOT(>=), NOT(<), NOT(<=), NOT(?),
  NOT(<>), NOT(<=>), NOT(?>), NOT(?>=),
  NOT(?<), NOT(?<=), NOT(?=)

- I have never seen a computer language that supports all of these 26 composite relations with 26 relational operators

Floating-Point

- Transreal arithmetic supports 7 composite relations:
  =, >, >=, <, <=, <>, <=>

- Transreal arithmetic supports negations of all of the composite relations:
  !=, !>, !>=, !<, !<=, !<>, !<=>

- Transreal arithmetic preserves symmetry (orthogonality) of negation that the IEEE standard breaks, this makes it easier to program with transreal numbers

Floating-Point

- 12 of the IEEE relations can raise an exception: >, >=, <, <=, <>, <=>, NOT(>), NOT(>=), NOT(<), NOT(<=), NOT(<>), NOT(<=>)
- Specifically, all of the relations that do not contain the predicate *unordered* can raise an exception on NaN
- None of the transreal relations can raise an exception so it is easier and safer to program with transreal numbers

Floating-Point

- The IEEE standard defines that $\mathrm{NaN}_i \neq \mathrm{NaN}_j$ so that it is false that $x = x$ for some floating-point objects, x
- The above breaks a cultural stereotype that everything is equal to itself and destroys equality in mathematical physics so that mathematical physics does not work with NaN

Floating-Point

- Transreal arithmetic has $x = x$ for all transreal numbers, x
- The above preserves a cultural stereotype that everything is equal to itself and maintains equality in mathematical physics

High Performance Computing

I have designed a new computer architecture and I have built a team to design, manufacture, test, and sell it!

- Pass tokens on a pipeline so that all cores can simultaneously receive and transmit tokens in every direction with zero Input/Output (I/O) latency
- Hold all working memory in on-chip cache, thereby accessing memory at processor speeds
- Run I/O from every edge of the chip with zero latency so that there is no memory wall

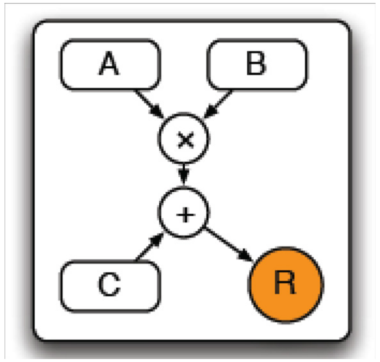
High Performance Computing

Hardware layouts say we can achieve:

- order 10 k double-precision floating-point cores on a chip
- order 10 kW per PFLOP of double-precision floating-point arithmetic

High Performance Computing

- The only arithmetical operation is $A \times B + C \rightarrow R$:



High Performance Computing

- All other arithmetical operations are synthesised from this one, because fabricating them would waste space in most of the processors on a chip
- The floating-point chip also has one non-arithmetical operation

High Performance Computing

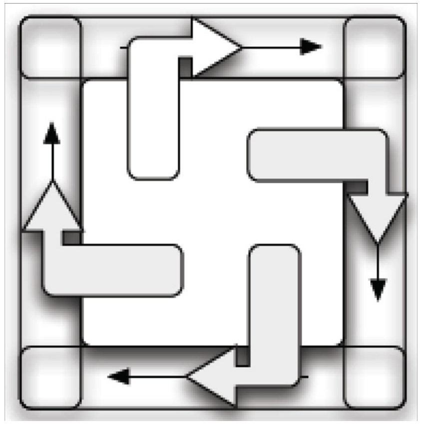
Fetchless architecture:

- Minimises circuitry
- Reduces on-chip fetch-latency to zero

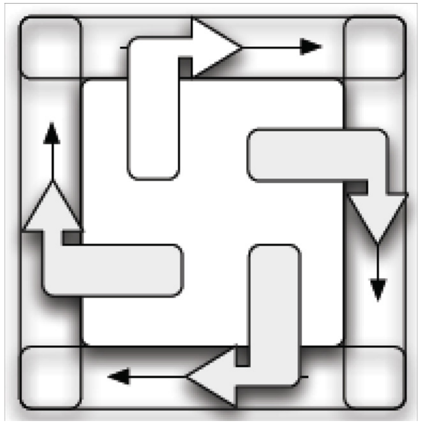
Achieved by token passing

High Performance Computing

- Tokens transmitted, conditionally on the sign of $A \times B + C \rightarrow R$, through every edge of the square processor and internally (Grey arrow blocked, white transmitted)

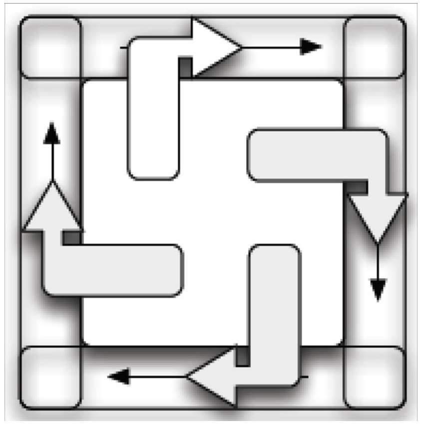


High Performance Computing



- There are four signs – negative, zero, positive, nullity – encoded in two sign bits so all selectors are maximally efficient

High Performance Computing



- If desired, each of the four separate signs can transmit a token in a different direction

High Performance Computing

There are no arithmetical error states so:

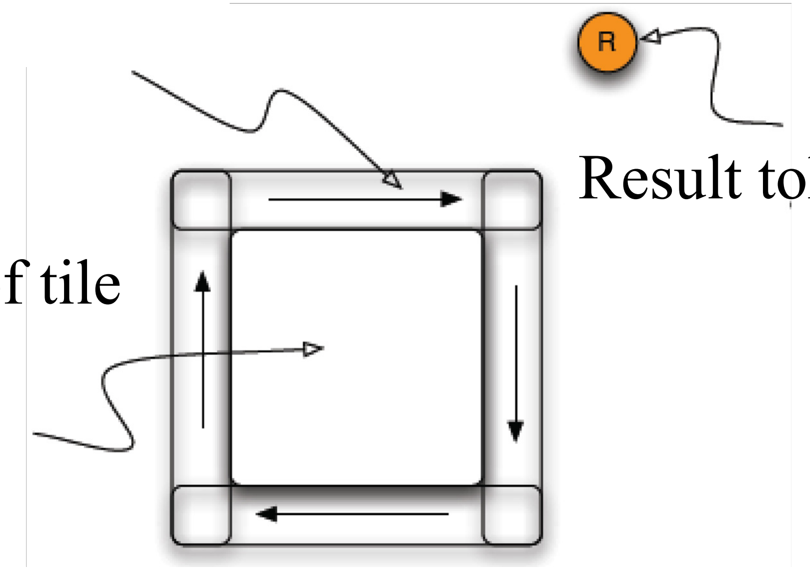
- There is no arithmetical error handling circuitry on a processor
- Program execution is more secure

Core Tile

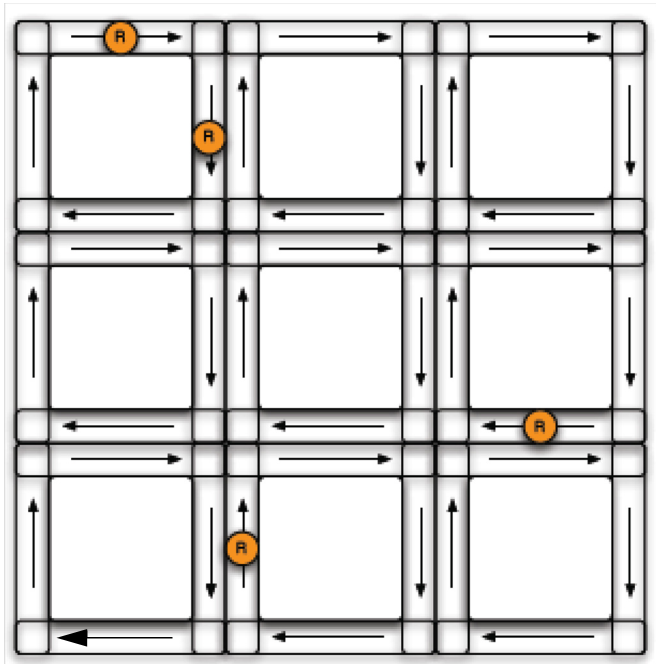
Pipeline
N, S, E, W

Result token

Processor in centre of tile



Pipeline

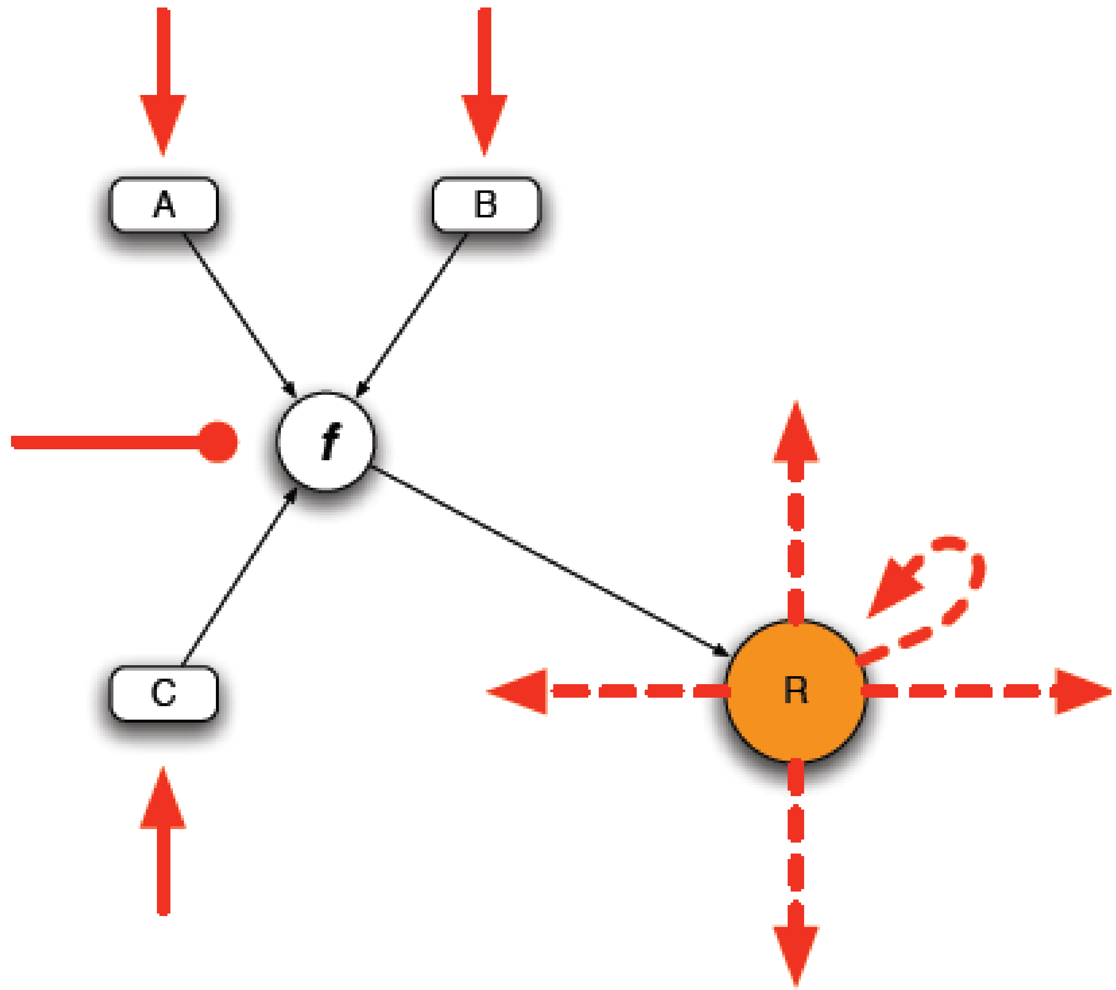


- Pipeline emerges from the processor tiles

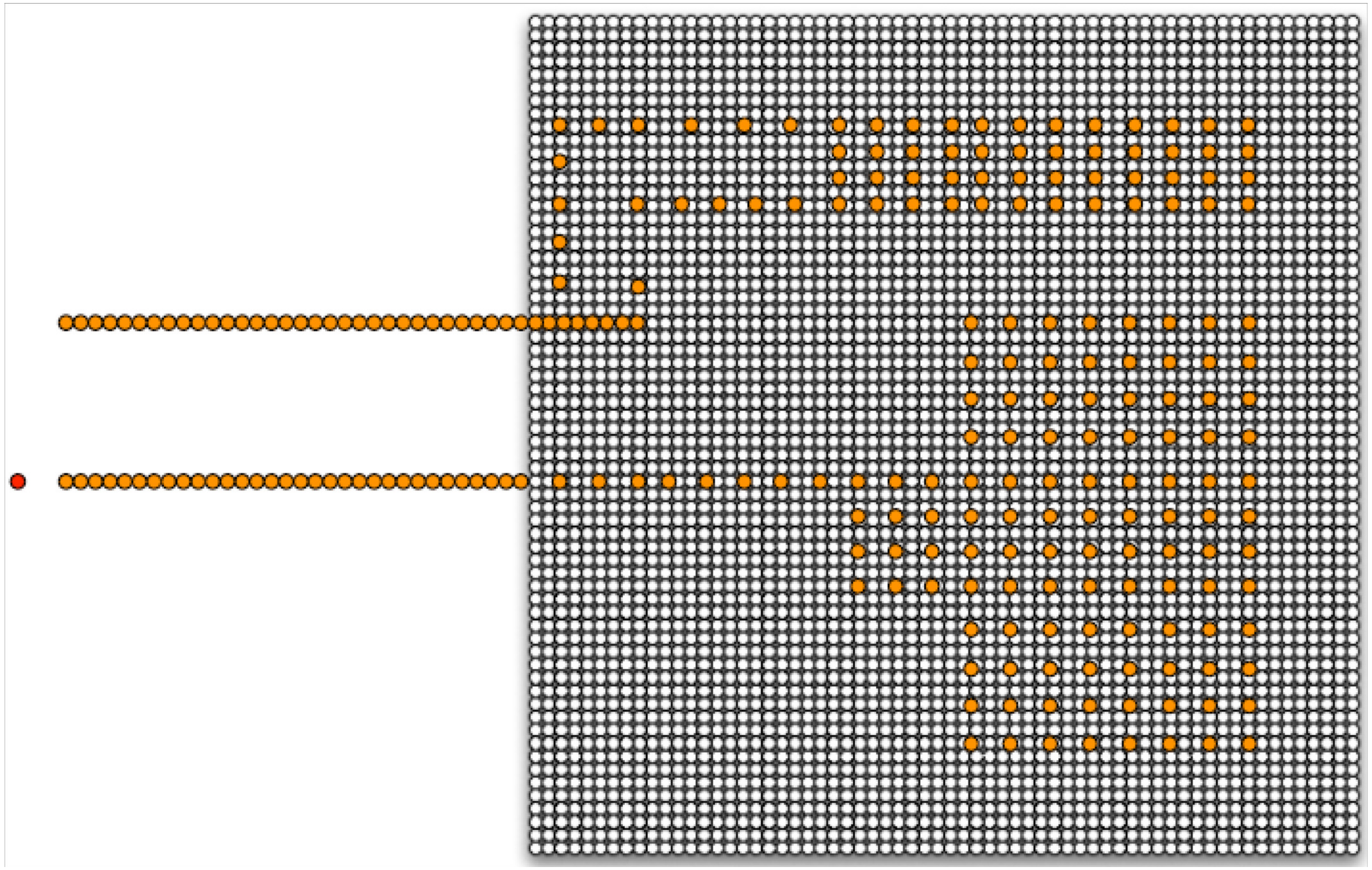
Pipeline

- Every core can simultaneously read from and write to the pipeline on every processor clock cycle with zero latency

Programming: an Army of Ants



Programming: Initial Orders



Programming: Is it Possible?

We have programmed these machines:

- Emulated a Turing complete machine
- Eliminated race conditions by using a single thread
- Eliminated race conditions by travel-time inequalities
- Implemented fully pipelined, mathematical functions with a throughput of one result per clock tick, and a latency down to half that of the Itanium 2 on: reciprocal, reciprocal square root, square root, exponential

Programming: Is it Possible?

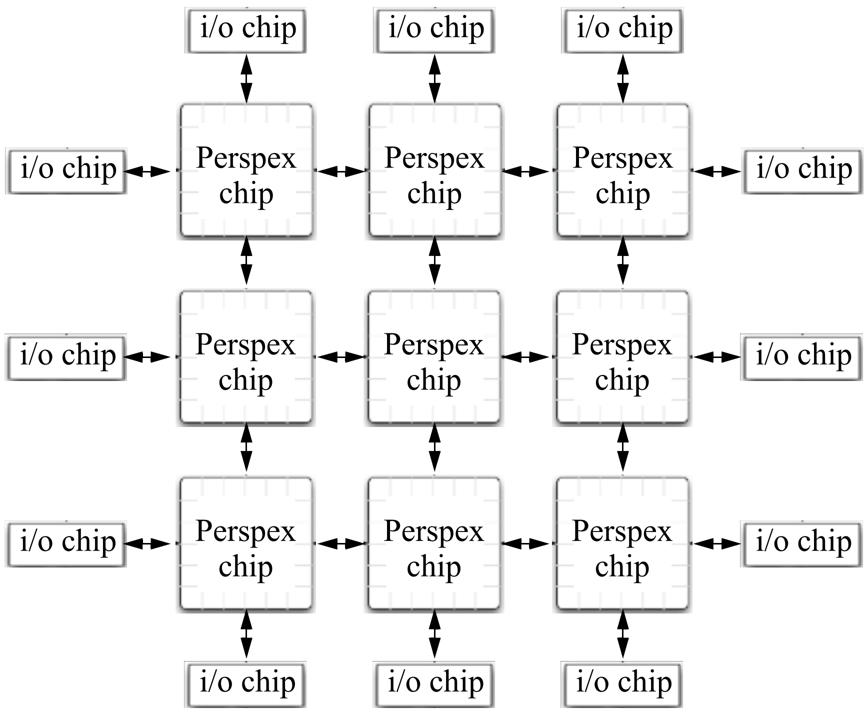
- Pipelines do not break on branches, they just tee-off in some city-block direction within the 2D surface of a chip
- Non-recursive subroutines have call and return implemented by branching so when they are in-lined they do not break pipelines
- Multiple calling points for a single subroutine introduce bubbles in each pipeline, and may break the pipeline – but code replication prevents this
- Loops force pipeline data into blocks of a size that will fit inside the loop, this breaks the pipeline unless the machine is huge or the problem is small

Compilation

- General compilation is straightforward, but uses our high performance computer as a co-processor
- Compilation via single assignment (functional programming) is attractive because of pipelining
- Systolic programming is highly applicable, but with a different I/O model
- Pipeline programming is highly applicable

I/O

- Chips can be tiled together with one pipeline input and one pipeline output per edge of the chip, each running at 250 M Tokens Per Second



I/O

- Chips have an address horizon, not an address space, so arbitrarily many chips can be tiled together
- Hence the performance of the architecture scales linearly

Performance

The current specification of the chip has:

- order 10 k cores
- cores clocked at 250 MHz
- 4 input channels, and 4 output channels, each running at 250 M Tokens Per Second
- power consumption of order 10 kW per PFLOP

Performance

- We have implemented conventional programs
- We have implemented systolic programs
- We have implemented pipelined programs
- We get better pipeline latencies than other architectures
- We get better pipeline throughput than other architectures
- We can sometimes match, but can never beat, systolic architectures

Performance

- It is not practical to implement *programmable* systolic arrays in ASIC, but our chip is a viable alternative
- Compiling by travel-time inequalities builds in some robustness to asynchronous operation of the array of processors
- Asynchronicity can be built into the array to smooth power usage
- Our power consumption is of order 1% of competing architectures

Offer

- We plan to sell high performance computers at 5 M US Dollars (USD) per PFLOP, three years after funding a start-up company
- We offer a discount of two, 1 PFLOP machines for 5 M USD for anyone willing to make staged payments of one third on contract, one third on tape out, and one third on delivery
- We plan to accept discounted orders for 5 to 10 PFLOPs
- If the project fails, we will return all unspent monies – this spreads the risk between customers

The Future

- Transmathematics and its application to transphysics will develop slowly
- Transcomputation can deliver benefits now
- The first company to sell a trans-floating-point chip will kill its competition on marketing strengths: more precision for the same bits and astronomically fewer error states
- The first company to deliver massively pipelined transreal computers will transform the market for high performance computing, signal processing, cryptanalysis, and so on …