Transreal Arithmetic and Analysis

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Dedication

This presentation is dedicated to the *USS Yorktown* which was stranded for 2 hours 40 minutes when a division-by-zero error crashed its entire network of computers, causing its engines to stop.

Overview of Transreal Arithmetic

- Transreal arithmetic is a total arithmetic – every operation of arithmetic can be applied to any number(s) and the result is a number.
- In particular, it allows division by zero.
- It appears to be different from all previous total arithmetics.
- It extends geometry and analysis.
- The biggest difference is in the treatment of nullity, $\Phi = 0/0$, as a fixed number.

Real Arithmetic

- The standard arithmetic of real numbers outlaws division by zero.
- Consequently many values of formulae are undefined.
- This is not acceptable to computer manufacturers or programmers. They use computer arithmetics that do allow division by zero.
- Therefore, standard arithmetic is *externally* (sociologically) *invalid.* It does not describe the arithmetic that people use in their daily lives as programmers or users of computer systems.

IEEE Floating-Point Arithmetic

The most popular computer-arithmetic that allows division by zero is IEEE floating-point arithmetic.

- It uses three objects that are not numbers: NaN, ±Inf.
- NaN is not a limit. It is an undefined object.
- ±Inf are defined as limits.
- It is an *internally valid* (consistent) model of arithmetic.
- It is an *externally invalid* model of arithmetic because NaN $\neq$ NaN, but equals means $x = x$ for all x.

IEEE Floating-Point Arithmetic

- IEEE floating-point arithmetic is dangerous.
- Changing any type to floating-point breaks the semantics of equality.

```
if x = y
then action_1
else action_2
endif
```

- This breaks a cultural stereotype. This is the most dangerous kind of behaviour it is possible to have in a man-machine interface.
- IEEE mandates that software contains this error.

Bottom

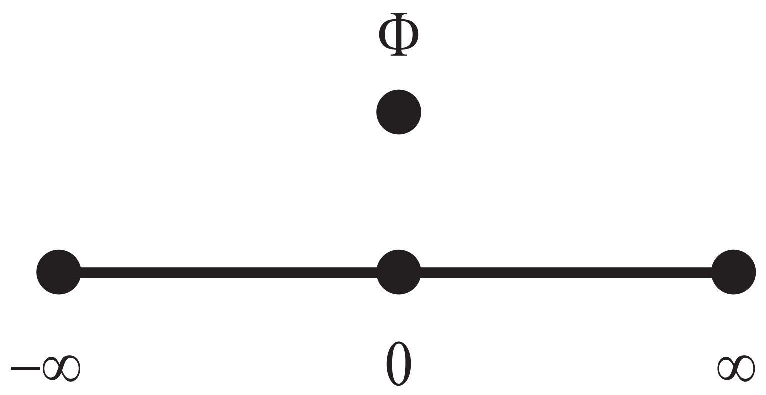
Any partial function can be made total by including the object bottom ($\perp$).

- Simply adding bottom to real arithmetic gives $-\infty = \infty = \Phi = \perp$.
- Adding $-\infty$, ∞ and bottom to real arithmetic gives $\Phi = \perp$, but this leaves considerable freedom to specify equality, ordering, and topology.
- It does not appear that any previous use of bottom exactly replicates transreal arithmetic, geometry, and analysis.

Transreal Arithmetic

Transreal arithmetic was first developed ten years ago.

- It is defined on a set of numbers including the three, fixed, numbers Φ, $\pm\infty$.
- Φ is a number, but it is not a limit.
- $\pm\infty$ are defined as numbers, but they can be limits.



Operations of Transreal Arithmetic

For all positive numbers k:

- $\infty = \dfrac{k}{0}$ and in least terms $\infty = \dfrac{1}{0}$
- $\Phi = \dfrac{0}{0}$ is in its least terms
- $-\infty = \dfrac{-k}{0}$ and in least terms $-\infty = \dfrac{-1}{0}$

Operations of Transreal Arithmetic

There are just three special rules in transreal arithmetic.

- Numbers are always reduced to least terms as soon as they are produced.
- $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
- $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

Many of the arithmetical algorithms we teach children are wrong when applied to transreal numbers, but they are easy to correct.

Transreal Arithmetic

- Real arithmetic has many side conditions to (unnecessarily) outlaw division by zero.
- Transreal arithmetic has less restrictive side conditions that do allow division by zero.
- Transreal arithmetic has 32 axioms, only 10 of them are new.
- Most of the new axioms are prefigured in IEEE floating-point arithmetic.

Axioms (Addition)

Additive Associativity: $a + (b + c) = (a + b) + c$ [A1]

Additive Commutativity: $a + b = b + a$ [A2]

Additive Identity: $0 + a = a$ [A3]

Additive Nullity: $\Phi + a = \Phi$ [A4]

Additive Infinity: $a + \infty = \infty : a \neq -\infty, \Phi$ [A5]

Axioms (Subtraction)

Subtraction as Sum with Opposite:
$a - b = a + (-b)$ [A6]

Bijectivity of Opposite: $-(-a) = a$ [A7]

Additive Inverse: $a - a = 0 : a \neq \pm\infty, \Phi$ [A8]

Opposite of Nullity: $-\Phi = \Phi$ [A9]

Non-null Subtraction of Infinity:
$a - \infty = -\infty : a \neq \infty, \Phi$ [A10]

Subtraction of Infinity from Infinity: $\infty - \infty = \Phi$ [A11]

Axioms (Multiplication)

Multiplicative Associativity:

$a \times (b \times c) = (a \times b) \times c$ [A12]

Multiplicative Commutativity: $a \times b = b \times a$ [A13]

Multiplicative Identity: $1 \times a = a$ [A14]

Multiplicative Nullity: $\Phi \times a = \Phi$ [A15]

Infinity Times Zero: $\infty \times 0 = \Phi$ [A16]

Axioms (Division)

Division: $a \div b = a \times (b^{-1})$ [A17]

Multiplicative Inverse: $a \div a = 1 : a \neq 0, \pm\infty, \Phi$ [A18]

Bijectivity of Reciprocal: $(a^{-1})^{-1} = a : a \neq -\infty$ [A19]

Reciprocal of Zero: $0^{-1} = \infty$ [A20]

Reciprocal of the Opposite of Infinity: $(-\infty)^{-1} = 0$ [A21]

Reciprocal of Nullity: $\Phi^{-1} = \Phi$ [A22]

Axioms (Ordering)

Positive: $\infty \times a = \infty \Leftrightarrow a > 0$ [A23]

Negative: $\infty \times a = -\infty \Leftrightarrow 0 > a$ [A24]

Positive Infinity: $\infty > 0$ [A25]

Ordering: $a - b > 0 \Leftrightarrow a > b$ [A26]

Less Than: $a > b \Leftrightarrow b < a$ [A27]

Greater Than or Equal: $a \geq b \Leftrightarrow (a > b) \vee (a = b)$ [A28]

Less Than or Equal: $a \leq b \Leftrightarrow b \geq a$ [A29]

Axioms (Quadrachotomy)

Quadrachotomy:

Exactly one of

$(a < 0)$, $(a = 0)$, $(a > 0)$, $(a = \Phi)$ [A30]

Axioms (Distributivity)

Distributivity:

$$a \times (b + c) = (a \times b) + (a \times c) :$$

$$\neg((a = \pm\infty) \wedge (\operatorname{sgn}(b) \neq \operatorname{sgn}(c)) \wedge (b + c \neq 0, \Phi))$$

[A31]

Axioms (Lattice Completeness)

Lattice Completeness:

The set, X, of all transreal numbers, excluding Φ, is lattice complete because

$$\begin{aligned} &\forall Y : Y \subseteq X \Rightarrow \\ &(\exists u \in X : (\forall y \in Y : y \leq u) \wedge \\ &\quad (\forall v \in X : (\forall y \in Y : y \leq v) \\ &\qquad \Rightarrow u \leq v)) \end{aligned} \qquad [A32]$$

Consistency

- A model of real arithmetic was constructed in the computer-proof system Isabelle/HOL.
- This model was extended to transreal arithmetic.
- The validity of each of the transreal axioms was then established mechanically for this extended model.

This proves that the axioms are:

- Self-consistent.
- Contain real arithmetic as a proper subset.

Dissolving Counter-Proofs

All of the standard counter-proofs which attempt to demonstrate that division-by-zero is impossible are:

- Falsified by the computer proof of the consistency of transreal arithmetic.
- Explicitly shown to be erroneous by hand proofs.

Algebraic Counter-Proofs

All number systems have $a \div b = a \times b^{-1}$, but:

- Standard arithmetics (algebras) define a^{-1} such that:

$$a \times a^{-1} = 1 : a \neq 0.$$

- Transreal arithmetic (algebra) defines a^{-1} such that:

$$a^{-1} = \frac{\text{denominator}(a)}{\text{numerator}(a)}$$

These are different definitions of division so the algebraic counter proofs are irrelevant.

Analytic Counter-Proofs

- Transreal arithmetic defines the signed infinities and nullity as fixed numbers:

$$-\infty = \frac{-1}{0}, \infty = \frac{1}{0}, \Phi = \frac{0}{0}$$

- Transreal arithmetic has no need to employ limits to define these numbers.
- Arithmetic is logically prior to analysis (calculus) so all analytical counter examples are irrelevant.

Geometrical Counter-Proofs

It seems that all geometrical counter proofs to division by zero are defeated by this fact:

- The point at nullity lies outside any space spanned by real vectors.

Offer to Dispel Counter Proofs

- Despite the fact that all published counter proofs to division by zero are irrelevant, I will dispel any particular counter proof you care to present to me.

Exponential Function

When negative exponents are written as explicit reciprocals the exponential function is well defined for all transreal $x \in R \cup \{-\infty, \infty, \Phi\}$.

$$\exp(x) = \begin{cases} (\exp(-x))^{-1} & : x < 0 \\ \lim\limits_{k \to \infty} 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots + \dfrac{x^k}{k!} & : \text{otherwise} \end{cases}$$

Whence $\exp(-\infty) = 0$, $\exp(\infty) = \infty$, $\exp(\Phi) = \Phi$.

Natural Logarithm

The transreal logarithm is well defined for all non-negative transreal numbers. In particular:

$$\ln \Phi = \ln e^{\Phi} = \Phi$$

$$\ln \infty = \ln e^{\infty} = \infty$$

$$\ln 0 = \ln e^{-\infty} = -\infty$$

$$\ln(-x) = \Phi : -x < 0$$

Hyperbolic Trigonometry

The hyperbolic trigonometric equations are defined for all transreal numbers.

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

Hyperbolic Trigonometry

$$\tanh x = \begin{cases} -1 & : x = -\infty \\ 1 & : x = \infty \\ \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} & : \text{otherwise} \end{cases}$$

Hyperbolic Trigonometry

$$\coth x = \begin{cases} -1 & : x = -\infty \\ \Phi & : x = 0 \\ 1 & : x = \infty \\ \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}} & : \text{otherwise} \end{cases}$$

Trigonometry

The trigonometric equations are defined for all transreal numbers. In particular:

$$\cos(\pm\infty) = \cos(\Phi) = \Phi$$

$$\sin(\pm\infty) = \sin(\Phi) = \Phi$$

$$\tan(\pm\infty) = \tan(\Phi) = \Phi$$

Trigonometric Identities

Trigonometric identities hold for all transreal x. In particular:

$$\cos^2 x + \sin^2 x = 1^x$$

$$\cosh^2 x - \sinh^2 x = 1^x$$

- Neither of these holds for real or IEEE floating-point arithmetic when $x = k/0$ with $k \neq 0$.
- In other words trigonometry does not work in IEEE floating-point arithmetic in almost all cases of division by zero.

Differential and Integral Calculus

It is conjectured that the whole of differential and integral calculus is well defined for all transreal numbers after adopting the definition:

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$$

In particular:

- Φ is an isolated point so it contributes zero to an area.
- Φ lies off the real-number line, extended with the infinities, so any definite integral with a bound of nullity contributes nullity to an area.

Mathematical Physics

- It is conjectured that when Maxwell's equations are derived in transreal numbers, there are no naked singularities (infinities affecting a neighbourhood of space).

- Similarly, it is conjectured that all naked singularities can be removed from all equations in mathematical physics by deriving them in transreal arithmetic.

- If so, this speaks to the physical reality of the transreal numbers $\pm\infty$, Φ.

Conclusion

Given the foregoing results it is easy to:

- Make programming languages and computers that cannot suffer a division by zero error nor any arithmetical exception.
- Make computers that carry out calculations much, *much* faster than existing computers.
- Give AI programs a direct link to the physical world without using any intermediate knowledge representations or transducers.
- Extend geometry and analysis so that they are total.

Question

- Under the laws of the USA, is the continued use of IEEE floating-point arithmetic negligent?

Challenge

The number 0^0 is not defined in real arithmetic, but in transreal arithmetic we have:

$$0^0 = 0^{(1-1)} = 0^1 \times 0^{-1} = \left(\frac{0}{1}\right)^1 \times \left(\frac{0}{1}\right)^{-1} = \frac{0}{1} \times \frac{1}{0} = \frac{0}{0} = \Phi$$

What makes this so terribly difficult that it was not derived earlier in the 1,200 years since the invention of zero?

Is it the last part of the equation, defining the fixed number nullity, $0/0 = \Phi$, that is so difficult?