Perspex Machine XI: Topology of the Transreal Numbers

James A.D.W. Anderson

Abstract – **The transreal numbers are a total number system in which every arithmetical operation is well defined everywhere. This has many benefits over the real numbers as a basis for computation and, possibly, for physical theories. We define the topology of the transreal numbers and show that it gives a more coherent interpretation of two's complement arithmetic than the conventional integer model. Trans-two's-complement arithmetic handles the infinities and 0/0 more coherently, and with very much less circuitry, than floating-point arithmetic. This reduction in circuitry is especially beneficial in parallel computers, such as the Perspex machine, and the increase in functionality makes Digital Signal Processing chips better suited to general computation.**

Index Terms – **Transmetric space, transreal number, two's complement.**

I. Introduction

The Perspex machine [1] was introduced as a theoretical computer that carries out all computations geometrically using transreal co-ordinates. Work is under way to develop a practical Perspex chip, but the results reported here also apply to conventional architectures.

We briefly review the transreal numbers and then present various topological definitions. We develop some useful, but elementary, results to make clear how the transreal numbers relate to the real numbers. We then show that two's complement arithmetic is a better model of transinteger arithmetic than of integer arithmetic.

The central issue is that two's complement arithmetic has two fixed codes that do not change under complement (negation). These are the codes for zero and for the most negative, represented, integer, n. Real arithmetic has just one number, the number zero, that does not change sign under negation so the two's complement behaviour $-n = n$ is erroneous in real arithmetic. By contrast, transreal numbers have a second number, nullity, that does not change sign. Mapping this number onto the second, fixed, code means that negation has its usual meaning everywhere. (See the Appendix.) This is beneficial, and removes a bug from two's complement arithmetic in which $abs(n) < 0$ when n is the most extreme integer. This pernicious bug is nearly universal in conventional computers [2] and can cause software to instruct a motion of the largest magnitude in the opposite direction from what was intended so its removal is highly beneficial.

Manuscript received October 4, 2007; revised January 4, 2008.

J.A.D.W. Anderson is in Systems Engineering at Reading University, Whiteknights, Reading, Berkshire, England, RG6 6AY (e-mail:j.anderson@reading.ac.uk)

Mapping the signed infinities, $\pm\infty$, to the most extreme two's complement codes, excluding nullity (see the Appendix), and saturating the arithmetic, so that unrepresentably large numbers round off to these infinities, produces an arithmetic with many of the exception handling properties of IEEE floating-point arithmetic [3], but with the much smaller circuitry associated with two's complement arithmetic. This reduction in circuitry [4] can be used to fabricate many more processors on a chip. It can also be used to make Digital Signal Processing (DSP) chips better suited to general programming, thereby expanding the number of tasks for which DSP chips can be used, bringing the advantages of low power, low cost, and high speed to these areas of computing.

In the next two subsections we introduce the transreal numbers and describe the structure of the paper, before engaging with the main material.

I.i Transreal Numbers

The transreal numbers are a generalisation of the *real numbers,* R. Firstly, the real numbers are extended with the signed infinities, $\pm\infty = (\pm 1)/0$, to give the *extended real numbers*, $R^E = R \cup \{-\infty, \infty\}$, as usual. Secondly, the extended real numbers are augmented with nullity, $\Phi = 0/0$, to give the *transreal numbers,* $R^T = R^E \cup \{\Phi\}$. The arithmetic of transreal numbers was recently axiomatised [5] and various real functions were extended to transreal functions by considering sums of infinite series [6]. We now put transreal analysis [6] on a firmer footing by developing the topology of transreal numbers. Amongst other things, this allows us to draw a sharp distinction between series that have unboundedly many terms, but no infinity term, which *asymptote to infinity,* from such series that additionally have an infinity term, thereby *arriving exactly at infinity.* The difference between the two can be demonstrated in a standard topological example (Mendelson [7], page 62) that shows that the infinite union of closed sets need not be closed. (Definitions of closure [7] are given later, but it is useful to present the example here.)

Consider closed sets corresponding to the closed interval $[1/n, 1]$ and take the union of infinitely many of these closed sets, indexed over the positive integers, $Z^+ = 1, 2, \ldots$, then $\bigcup_{n \in Z^+} [1/n, 1] = (0, 1]$ is not a closed set [7].

But now take an infinite union over all of the extended, positive integers, $Z^{+E} = Z^{+} \cup \{\infty\}$, then $\bigcup_{n \in Z^{+E}} [1/n, 1] = [0, 1]$ is closed because $1/\infty = 0$.

This example serves as a warning that a series, sum or limit can give different results depending on whether it is taken asymptotically to infinity or exactly to infinity. This is not a surprise, except that the results at infinity can be calculated arithmetically. A further example, based on the Dirac Delta, $\delta(t)$, might make this clear.

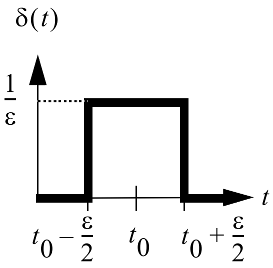

Fig. 1: Dirac Delta as the asymptote when epsilon tends to zero

Consider a box function, Fig. 1, which is zero everywhere, except in a neighbourhood around zero. For some positive epsilon, ε, the function has a step at $t_0 - \varepsilon$, rising to a height of $1/\varepsilon$, and has a step at $t_0 + \varepsilon$, falling to a height of zero. The area under the graph is the area of the rectangle with width, $w = \varepsilon$, and height $h = 1/\varepsilon$. That is, the area is $a = w \times h = \varepsilon/\varepsilon$. Suppose that epsilon tends *asyptotically* to zero, $\varepsilon \to 0$. Then the width tends *asymptotically* to zero, $w \to 0$, the height tends *asymptotically* to infinity, $h \to \infty$, and the area is everywhere *equal* to unity, $a = w \times h = \varepsilon/\varepsilon = 1$, because epsilon is everywhere a real number, greater than zero. In this *asymptotic* limit, $\varepsilon \to 0$, the box function is the Dirac Delta [8]. But now calculate the extreme value transarithmetically [5]. When epsilon is *exactly* zero, $\varepsilon = 0$. We have width *exactly* zero, $w = \varepsilon = 0$, height *exactly* infinity $h = 1/\varepsilon = 1/0 = \infty$ whence the area is *exactly* nullity, $a = w \times h = 0 \times \infty = (0/1) \times (1/0) = (0 \times 1)/(1 \times 0) = 0/0 = \Phi = 0/0 = \varepsilon/\varepsilon$. This *is not* the Dirac Delta, but it does demonstrate that the extreme value can be calculated transarithmetically at an instant, that is, without regard to limiting processes. In the Discussion we take up the question of the physical meaning of this extreme configuration of the box function, but we are more immediately concerned to develop the topology of the transreal numbers.

Perhaps we have laboured the very well known point that the asymptote of a function at a number need not be equal to the value of the function at that number, but, in this paper, we are obliged to mix asymptotes and values so it is important that the reader has the mathematical and notational distinction in mind. Everywhere in this paper, $-\infty$, ∞, and Φ are strictly transreal numbers [5]. They *are not* indefinite or undefined. They are *fixed* numbers and *do not* represent a range of values. Asymptotic limits are shown using the arrow notation, $a \to b$, and exact values are shown by equality $a = b$.

The figure below may help the reader. It shows zero at the mid-point of the real-number line. To the right of the line are points labelled with aleph, $\aleph_i$, these points represent transfinite numbers of some kind, such as Cantor's transfinite numbers [9], Robinson's unlimited numbers [10], [11], Conway's infinite surreal numbers [12], or various others. The point that lies to the extreme right is the number omega, Ω, it is the largest infinity. If the continuum hypothesis [9] is taken to be true then $\aleph_i$ are discrete points, but if it is taken to be false then the $\aleph_i$ may lie on a line. Even so, it is an open question whether the real-number line, this putative line at infinity, and omega are connected. The negative transfinite numbers and negative omega are shown to the left of the real-number line. Nullity, Φ, is shown off the line.

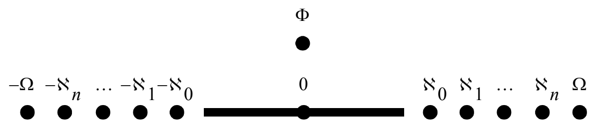

Fig. 2: The transreal numbers augmented with some transfinite numbers, $\aleph_i$

Nullity is conventionally drawn above zero, but its position in the diagram is arbitrary, so long as it does lie off the real-number line and the projection of that line through the signed, transfinite numbers to signed omega. (Compare with the Appendix.) The infinitesimal numbers [10], [11], [12] lie in a small region around zero, but they are not marked on the diagram.

In this paper we are not concerned with the transfinite numbers, $\aleph_i$. Though, if we wished to, we could extend the transreal numbers to include transfinite and infinitesimal numbers in much the same was as the real numbers have been extended [10], [11], [12]. Here we are concerned only with the real numbers, and the three *strictly transreal* numbers $-\infty = -\Omega$, $\infty = \Omega$, and Φ. The transfinite numbers, $\aleph_i$, are shown only to motivate the assumption that the infinities, $\pm\infty = \pm\Omega$, are disconnected from the real-number line. We further assume that the signed infinities are disconnected from each other and from nullity, and that nullity is disconnected from every other number. Thus our numbers fall into four separate pieces: the real-number line, $R = (-\infty, \infty)$; the point at minus infinity, $\{-\infty\} = [-\infty, -\infty]$; the point at infinity, $\{\infty\} = [\infty, \infty]$; and the point at nullity, $\{\Phi\} = [\Phi, \Phi]$. As usual, (a, b) is the *open interval* $\{x : a < x < b\}$ and $[a, b]$ is the *closed interval* $\{x : a \le x \le b\}$.

As expected, there is an intermediate value between any real number and infinity. Because, given an arbitrary real number, a, we may choose a real number $i > a$, but all real numbers are less than infinity [5] so, in particular, $i < \infty$ is an intermediate value. Similarly, there is an intermediate value between any real number and minus infinity. However, nullity is not ordered [5] so there is no intermediate value between nullity and any other number. One consequence of this is that we can define infinite limits, as usual, by functions that grow without a real bound. But as there is no intermediate value between nullity and any other number, the distance from any number *to* nullity cannot be reduced, and growing unboundedly *is not* moving in the direction of nullity. As no function can approach nullity, it follows that no limiting process can take on the value nullity, so nullity is not the limit of any function. A function may, however, take on the value nullity. In general, it takes on this value at a discontinuity. For example $f(x) = x/(\sin x)$ is real in the neighbourhood of $x = 0$, but jumps discontinuously to nullity at $x = 0$. See [6].

I.ii Structure of the Paper

In the next section we introduce a topological space with the connectivity we have just presented. This space is not compact, because it contains the real-number line as a piece, but it does support the Heine-Borel theorem [13] over the real numbers, and an extension of it over the strictly transreal numbers. We also show that the constant function, $f(x) = \Phi$, is continuous, despite the fact that it does not have a limit. The constant functions $f(x) = -\infty$ and $f(x) = \infty$ are also continuous, and have the limits minus infinity and infinity, respectively.

In the subsequent section we show that the usual extension of metric spaces to include infinity is valid for transreal infinity, and we generalise metrics to *transmetrics* by replacing the symbol *greater-than-or-equals* with the symbol *not-less-than* in the definition of a metric. Here, *not-less-than,* $x \nless y$, means x is not less than y. This can be re-written in terms of intervals, $\{x : x \nless y\} = \{x : x \geq y\} \cup \{\Phi\} = [y, \infty] \cup [\Phi, \Phi]$, as is clear from the discussion and figure above. It should be noted very carefully that we do not make any other change during the definition of the transmetrics. The whole machinery of metric spaces is adopted as is. This allows us to adopt the usual definition of limits without change. A side effect of this is that it removes all indeterminate and undefined values from analysis. This seems to pose a mental block for some people who have become familiar with the idea of undefined and indeterminate values and rely on these to circumvent the non-total nature of, say, real arithmetic. But the situation is very clear: analysis defines the limits of various general functions, for example, those that converge to a real value from below, those that converge to a real value from above, those that grow unboundedly large, and so on [14]. Specific functions may, or may not, have limits. If the specific functions do have limits then the limits are fixed numbers. Conversely, if the specific functions do not have limits at a point, or more generally, then there is no limit at that place. It is an abuse of language, albeit a common one, to say that a limit which does not exist takes on an *indefinite* or *undefined* value. As a matter of plain speaking, we should rather say that a limit which does not exist does not take on any value at all. We take up this issue again in the Discussion.

We end with some remarks about how the results obtained here generalise mathematics, how they might apply to physical theories, and how they do apply to the design and programming of digital computers.

II. Topological Spaces

In this section we make some tutorial remarks about topological spaces for the benefit of programmers and hardware designers who require a clear understanding of the relationships that hold amongst the transreal numbers. We then derive various elementary, topological, properties of these numbers.

II.i Definition of a Topological Space

In this section we present the standard definition of a topological space.

The equation $S = \langle P, T\rangle$ identifies a *topological Space, S*, with the tuple $\langle P, T\rangle$. Here, P is a non-empty set of *Points* and T is a *Topology*, being a set of subsets of P, each of which subsets is defined to be an *open set.* Note that T may, or may not, be the set of all subsets of P.

A topological space obeys five axioms, though these are usually set out textually as fewer axioms [7], [13], [15], [16], [17], [18].

The empty set, $\varnothing$, is a member of T. (A1)

The set of points, P, is a member of T. (A2)

Every member of T is *open.* (A3)

The intersection of finitely many members of T is a member of T, and hence is open. (A4)

The union of finitely or infinitely many members of T is a member of T, and hence is open. (A5)

A subset, X, of P is said to be *closed* if its complement, $P - X$, is a member of T. That is, if its complement in P is open. The terms open and closed are not mutually exclusive. A set may be open, closed, open and closed, or neither open nor closed. A set which is both open and closed is called *clopen,* but there is no name for a set that is neither open nor closed.

II.ii Topological Space of Transreal Arithmetic

In this section we define a specific topology which, we hypothesise, is consistent with transreal arithmetic. In later sections we show that various transreal generalisations of metric spaces are consistent with this topology.

Note that *A*xioms are shown by (An), *D*efinitions by (Dn), *T*heorems by (Tn) and equations simply by their number (n).

We take the set of points, $P = R^T$, as the full set of transreal numbers [5]. (A6)

Every open interval of the real numbers, (a, b), is a member of T, and hence is open. (A7)

The minus infinity interval, $[-\infty, -\infty]$, is a member of T, and hence is open. (A8)

The infinity interval, $[\infty, \infty]$, is a member of T, and hence is open. (A9)

The nullity interval, $[\Phi, \Phi]$, is a member of T, and hence is open. (A10)

Note that the nullity interval is $\{x : \Phi \le x \le \Phi\} = \{\Phi\}$. This is because nullity is unordered, so the *less-than* part of the ordering relationship, *less-than-or-equals,* does not hold, leaving just $x = \Phi$. Consequently, this is the only interval that has nullity as a bound.

The interval $[-\infty, \infty]$ is a member of T, and hence is open, because $[-\infty, \infty] = \{-\infty\} \cup R \cup \{\infty\}$ is a union of open intervals, and hence is open. (T1)

II.iii Connectivity

In this section we show that the transreal numbers fall into four distinct pieces: $\{-\infty\}$, R, $\{\infty\}$, $\{\Phi\}$.

The connectivity of topological spaces can be defined in several ways. One way is via a topological partition [13]. A partition $A \mid B$ of a set of points P is a pair of non-empty sets of points A, B such that $A \cup B = P$, $A \cap B = \varnothing$, and both A and B are open. It follows that A is closed because its complement, $P - A = B$, is open. Similarly, B is closed because its complement, $P - B = A$, is open. Hence A, B are both clopen. A topological space, S, is said to be connected if and only if its underlying set of points, P, admits no partition.

Now $[\Phi] \mid [-\infty, \infty]$ is a partition because $[\Phi] \cup [-\infty, \infty] = R^T = P$, $[\Phi] \cap [-\infty, \infty] = \varnothing$, the sets $[\Phi], [-\infty, \infty]$ are non-empty, and both $[\Phi]$ and $[-\infty, \infty]$ are open (A10), (T1). In other words, the topological space implied by transreal arithmetic falls into two pieces: the extended, real-number line, $[-\infty, \infty]$, and the point at nullity, $[\Phi]$. The point at nullity cannot be partitioned because it is a singleton set, but the extended number line can be partitioned further. Firstly, $[-\infty, -\infty] \mid R \cup \{\infty\}$ so that the point at minus infinity, $[-\infty, -\infty] = \{-\infty\}$, is a discrete piece. Secondly, $\{-\infty\} \cup R \mid [\infty, \infty]$ so that the point at infinity, $[\infty, \infty] = \{\infty\}$, is a discrete piece. The proofs are similar to the proof just given. The remaining piece, the real-number line, $(-\infty, \infty)$, cannot be partitioned further, as is very well known. The proof is easy. Thirdly, if we attempt the partition $(-\infty, c) \mid [c, \infty)$ with $c \in R$, we find that $[c, \infty)$ is not open so the partition fails. Fourthly, the attempted partition, $(-\infty, c] \mid (c, \infty)$ with $c \in R$, fails similarly. Thus, the topological space implied by transreal arithmetic falls into exactly four pieces: the point at nullity, $\{\Phi\}$, the point at infinity, $\{\infty\}$, the point at minus infinity, $\{-\infty\}$, and the real-number line, $(-\infty, \infty)$. In fact, we specified the above topology, (A6) – (A10), so that it would have exactly this connectivity.

II.iv Compactness

In this section we adopt the standard definition of a compact topological space and show that our topological space is not compact precisely because it contains the real-number line as a distinct piece.

A *Cover* of an arbitrary set A is a collection, C, of sets such that $A \subset \bigcup_{c \in C} C$. A cover is said to be *finite* or *infinite* as its cardinality is finite or infinite. A *Subcover*, S, of a given cover, C, is a subcollection $S \subset C$ which still forms a cover of A. If A is a subset of the points, P, in a topological space, and C is a cover of A, and every element of C is open in P then C is an *open cover* of A. A topological space, $\langle P, T\rangle$, is *compact* if every open cover of P has a finite subcover. A topological space, $\langle Q, U\rangle$, is a subspace of the topological space, $\langle P, T\rangle$, if $Q \subset P$ and $U \subset T$. Compare with Sutherland [13].

Let $\langle P, T\rangle$ be our topological space defined in Section II.ii. This space falls into four distinct parts: $\{-\infty\}$, R, $\{\infty\}$, $\{\Phi\}$. The sets $\{-\infty\}$, $\{\infty\}$, $\{\Phi\}$ are open, singleton sets so they have a finite subcover by identity. The remaining part, R, is open, but the open cover of R does not have a finite subcover and, therefore, is not compact [13]. Hence our topological space is not compact.

Whereas our topological space is not compact, it does contain the real-number line as a distinct piece. Hence, all compactness results that hold for R continue to hold for R^T where the strictly transreal points $-\infty, \infty, \Phi$ are a finite support for the extension of the theorem. In particular, the Heine-Borel theorem [13] holds: any closed, bounded interval, $[a, b]$, in R is compact. Notice that, as has just been shown, we also have that, $[-\infty, -\infty]$, $[\Phi, \Phi]$, and $[\infty, \infty]$ are compact. But no other intervals in R^T are compact. Firstly, $[-\infty, \infty] = \{-\infty\} \cup R \cup \{\infty\}$, but R does not have a finite subcover so it is not compact. Secondly, all intervals, with one bound a signed infinity and the other bound a real number, partition into the singleton set containing the signed infinity and an open interval of the real-number line that is not closed, and hence is not compact. This exhausts the intervals of R^T and completes the extension of the Heine-Borel theorem.

II.v Continuity

In this section we adopt the standard definition of continuity in a topological space and show that the total, constant function $f(x) = \Phi$ is continuous. Similarly, the functions $f(x) = -\infty$ and $f(x) = \infty$ are continuous.

Given two topological spaces, $S_1 = \langle P_1, T_1\rangle$ and $S_2 = \langle P_2, T_2\rangle$, and a map $f : P_1 \to P_2$, we say that f is *continuous* (with respect to the topologies T_1, T_2) if $U \in T_2 \Rightarrow f^{-1}(U) \in T_1$. Compare with Sutherland [13].

Let $S_1 = \langle P_1, T_1 \rangle$ be our topological space over the transreal numbers, Section II.ii. In particular, $P_1 = R^T$. Let $S_2 = \langle P_2, T_2 \rangle$ be the topological space with $P_2 = \{\Phi\}$ and $T_2 = \{\Phi\} \cup \{\varnothing\}$. Now, $f: P_1 \to P_2$ is the total, constant function $f(x) = \Phi$ for all transreal x in P_1. First, if $U = \{\Phi\}$ then $U \in T_2$ and $f^{-1}(U) = R^T \in T_1$. Secondly, the trivial case, if $U = \{\varnothing\}$ then $U \in T_2$ and $f^{-1}(U) = \varnothing \in T_1$. This completes the proof that f is continuous. Similarly, the functions $f(x) = -\infty$ and $f(x) = \infty$ are continuous. The question of continuity is taken up again in Sections V and VI.

III. Metric Spaces

In this section we make some tutorial remarks about metric spaces for the benefit of programmers and hardware designers who require a clear understanding of the relationships that hold amongst the transreal numbers. We then derive various elementary, topological properties of these numbers and show that they are consistent with the topological space defined above.

III.i Definition of a Metric Space

The equation $S = \langle P, m \rangle$ identifies a *metric Space*, S, with the tuple $\langle P, m \rangle$. Here, P is a non-empty set of *Points* and m is a *distance Metric*. A distance metric obeys four axioms, though these are sometimes set out as fewer axioms [7], [13], [15], [17], [18]. Patterson [18] reduces the number of axioms to just two.

The following axioms, (A11) – (A14), are defined for all a, b where $a, b \in P$.

$$m(a, b) = m(b, a) \quad \text{(A11)}$$

$$m(a, b) \geq 0 \quad \text{(A12)}$$

$$m(a, b) = 0 \Leftrightarrow a = b \quad \text{(A13)}$$

$$m(a, b) + m(b, c) \geq m(a, c) \quad \text{(A14)}$$

Distance metrics usually take on real values, but we allow distance metrics to take on transreal values.

III.ii Metric Spaces Defined on Transreal Numbers

Historically, [18], [19] metric spaces were developed as a formalisation of distance in Euclidean space so we should not be surprised that the strictly transreal numbers do not obey the axioms of metric spaces. Nonetheless, there are various ways to define a function that maps strictly transreal distances onto real distances and infinity so that the axioms are obeyed. One such method follows.

Let $m(a, b)$ be a distance metric and define a *Generalisation* $g(a, b)$ of it as follows:

$$g(a, b) = \begin{cases} 0 : a = b & (1) \\ m(a, b) : a \neq b \text{ and } m(a, b) \in R & (2) \\ \infty : \text{otherwise} & (3) \end{cases} \quad \text{[D1]}$$

Branch (1) and the guarding clauses on (2) and (3) together implement (A13). Branch (2) implements all of (A11) – (A14) for non-zero, real distances. Zero distances are handled by branch (1). Branch (3) maps all strictly transreal distances onto infinity. This branch is consistent with (A11) – (A13). It remains to be shown that (A14) holds in the presence of some strictly transreal a, b, c. There are 27 permutations to consider, depending on whether a, b, c take on the value r, ∞, Φ – where r is an arbitrary, real number. These permutations are tabulated below. The tables demonstrate that (A14) holds.

Table. 2 Real values of the generalised metric

$c = r_c$	r_b	∞	Φ
r_a	$m(r_a, r_b) + m(r_b, r_c) \geq m(r_a, r_c)$	$\infty \geq m(r_a, r_c)$	$\infty \geq m(r_a, r_c)$
∞	$\infty \geq \infty$	$\infty \geq \infty$	$\infty \geq \infty$
Φ	$\infty \geq \infty$	$\infty \geq \infty$	$\infty \geq \infty$

Table. 3 Strictly transreal values of the generalised metric

$c = \infty, \Phi$	r_b	∞	Φ
r_a	$\infty \geq \infty$	$\infty \geq \infty$	$\infty \geq \infty$
∞	$\infty \geq \infty$	$\infty \geq \infty$	$\infty \geq \infty$
Φ	$\infty \geq \infty$	$\infty \geq \infty$	$\infty \geq \infty$

It may be helpful to illustrate this generalisation with a few examples. Let $m(a, b) = \sqrt{(a-b)^2}$ be the Euclidean distance metric. Consider its generalisation $g(\infty, \infty)$. Clause (1) of (A1) immediately gives $g(\infty, \infty) = 0$. Now consider $g(-\infty, \infty)$. Clause (1) does not apply, because $-\infty \neq \infty$. If we regard $m(a, b)$ as a real function then it has no value for any $a, b = -\infty, \infty, \Phi$ so clause (2) does not apply. Alternatively, if we interpret the text of the Euclidean distance metric, "$m(a, b) = \sqrt{(a-b)^2}$," as being defined in terms of transreal variables and transreal operations, rather than real variables and real operations, then we have $m(-\infty, \infty) = \sqrt{((-\infty) - \infty)^2} = \sqrt{-\infty^2} = \sqrt{\infty} = \infty \notin R$ so, again, clause (2) does not apply. This leaves the default clause (3), giving $g(-\infty, \infty) = \infty$. Similarly $g(\infty, \Phi) = \infty$. On examining cases we find that the distance from any number, be it real or transreal, to itself is zero, but the distance from any strictly transreal number to any distinct number, whether real or strictly transreal, is infinity.

The advantage of this approach is that it can be used to generalise any standard metric to one that works on transreal numbers. Any mathematical procedure that depends on a metric space is generalised by this act. For example, real analysis is generalised to a transreal analysis by this act, though we *do not* take that approach here. Instead, we introduce a generalisation of metric spaces in Section IV, and use that to obtain a transreal analysis, thereby extending earlier work on transreal analysis [6].

III.iii Epsilon Neighbourhoods

In this section we translate the above family of metric spaces, generated by g, into a topological space by the procedure of metrisation [13]. In this procedure the set of points in the metric space is set equal to the set of points in the topological space so that both spaces share the same points. Then the open sets of the metric space are obtained via the distance metric and are set equal to the open sets of the topological space so that both spaces share the same open sets.

The open sets of a metric space are defined in terms of epsilon neighbourhoods. We adopt the standard definition [13] without change. Given a metric space $S = \langle P, m \rangle$, a point $p \in P$, and a strictly positive, real number, ε, the *open* ε*-ball neighbourhood* of p in S is the set: $B_\varepsilon(p) = \{x \in P : m(x, p) < \varepsilon\}$. We then adopt the standard definition of an *open set* without change [13]. A subset Q of P is *open in* P if, given any q in Q, there exists some $\varepsilon(q) > 0$ such that $B_{\varepsilon(q)}(q) \subset Q$. The values of epsilon, $\varepsilon(q)$, may be distinct for distinct points q in Q.

This is just the standard definition of an open set in a metric space. It remains only to consider the open sets at nullity and the infinities. We have: $B_\varepsilon(-\infty) = \{-\infty\}$, $B_\varepsilon(\infty) = \{\infty\}$, and $B_\varepsilon(\Phi) = \{\Phi\}$. Thus, these singleton sets are open in our family of metrics and, as $-\infty$, ∞ and Φ are distinct, these three sets are disconnected from each other and from all of the open sets in the real-number line, as in our topological space above. In other words, our family of metrics, generated by g, gives the same topology as our topological space.

IV. TRANSMETRIC SPACES

In this section we define transmetric spaces as a superset of metric spaces by admitting a distance of nullity.

IV.i Definition of a Transmetric Space

We say that the equation $S = \langle P, t \rangle$ identifies a *transmetric Space,* S*,* with the tuple $\langle P, t \rangle$. Here, P is a non-empty set of *Points* and t is a *Transmetric,* that is, a transreal distance function. We say that a transmetric obeys the four axioms (A15) – (A18). These axioms are defined for all a, b where $a, b \in P$. These are just the axioms of metric spaces, (A11) – (A14), with greater-than-or-equals, $\geq$, replaced by not-less-than, $\nless$. This substitution admits a distance of nullity. Nullity is equal to itself, but it is not ordered, so it is not less than, not equal to, and not greater than any other transreal number [5].

$$t(a, b) = t(b, a) \tag{A15}$$

$$t(a, b) \nless 0 \tag{A16}$$

$$t(a, b) = 0 \Leftrightarrow a = b \tag{A17}$$

$$t(a, b) + t(b, c) \nless t(a, c) \tag{A18}$$

When the distances are real or infinity the axioms of transmetric spaces are identical to the axioms of metric spaces, they differ only for the strictly transreal distance nullity.

IV.ii Example

An example might help to make the axioms clearer. We consider the triangle inequality, (A18) not (A14), over the Euclidean transmetric, t.

$$t(a, b) = \begin{cases} 0 : a = b \\ \sqrt{(a-b)^2} : \text{otherwise} \end{cases} \tag{D2}$$

As before, there are 27 permutations to consider, depending on whether a, b, c take on the value r, ∞, Φ – where r is an arbitrary, real number. These permutations are tabulated below. The tables demonstrate that (A18) holds for the Euclidean transmetric and show why it is necessary to replace *greater-than-or-equals* with *not-less-than.* For example, $\Phi \nless \infty$, is true, but $\Phi \geq \infty$ is false.

Table. 4 Transmetric with a real limb c

$c = r_c$	r_b	∞	Φ
r_a	$t(r_a, r_b) + t(r_b, r_c) \nless t(r_a, r_c)$	$\infty \nless t(r_a, r_c)$	$\Phi \nless t(r_a, r_c)$
∞	$\infty \nless \infty$	$\infty \nless \infty$	$\Phi \nless \infty$
Φ	$\Phi \nless \Phi$	$\Phi \nless \Phi$	$\Phi \nless \Phi$

Table. 5 Transmetric with an infinite limb c

$c = \infty$	r_b	∞	Φ
r_a	$\infty \nless \infty$	$\infty \nless \infty$	$\Phi \nless \infty$
∞	$\infty \nless \infty$	$\infty \nless \infty$	$\Phi \nless 0$
Φ	$\Phi \nless \Phi$	$\Phi \nless \Phi$	$\Phi \nless \Phi$

Table. 6 Transmetric with a nullity limb c

$c = \Phi$	r_b	∞	Φ
r_a	$\infty \nless \infty$	$\infty \nless \infty$	$\Phi \nless \Phi$
∞	$\infty \nless \infty$	$\infty \nless \infty$	$\Phi \nless \Phi$
Φ	$\Phi \nless \Phi$	$\Phi \nless \Phi$	$\Phi \nless \Phi$

IV.iii Epsilon Neighbourhoods

The transmetrics are identical to the metrics for all real numbers so the real numbers of a transmetric space metrise to our topology. It remains only to consider the open sets at nullity and the infinities. We have: $B_\varepsilon(-\infty) = \{-\infty\}$, $B_\varepsilon(\infty) = \{\infty\}$, and $B_\varepsilon(\Phi) = \{\Phi\}$. Thus, these singleton sets are open in our family of metrics and, as $-\infty$, ∞ and Φ are distinct, these three sets are disconnected from each other and from all of the open sets in the real-number line, as in our topological space above. In other words, our family of transmetrics gives the same topology as our topological space. But transmetrics differ from the generalisation of metrics in Section III.ii in that transmetrics admit a distance of nullity and the generalised metrics do not. Each is a valid basis for analysis, but we prefer to develop the analysis of transmetrics.

V. Intermediate Values And Limits

In order to consider limits of transreal functions we must use a transmetric, such as the transabsolute value, $|t| = \mathrm{sgn}(t) \times t$. See [5] for a discussion of the transreal sign function, *sgn*. But whichever transmetric we choose, all existing definitions of limits hold without change. Compare the following with [14].

$\lim_{x \to a} f(x) = l$ if for every real $\varepsilon > 0$ there is some real $\delta > 0$ such that, for all real x, if $0 < |x - a| < \delta$, then $|f(x) - l| < \varepsilon$. [D3]

$\lim_{x \to \infty} f(x) = l$ if for every real $\varepsilon > 0$ there is some real N such that, for all real $x > N$, it is the case that $|f(x) - l| < \varepsilon$. [D4]

$\lim_{x \to \infty} f(x) = \infty$ if for every real $\varepsilon > 0$ there is some real N such that, for all real $x > N$, it is the case that $f(x) > \varepsilon$. [D5]

Note that infinity and minus infinity may be the limits of functions, but that nullity is not the limit of any function. There is no intermediate value between nullity and any other number so it cannot be approached by any means, let alone by a limiting process. Nonetheless a function can take on the value nullity, regardless of the non-existence of a limit of nullity.

In section II.v, above, we show that the constant functions $f(x) = -\infty$, $f(x) = \infty$ and $f(x) = \Phi$ are continuous. This last observation is delicate. As usual, [14] a function f is continuous at a if its limit is equal to $f(a)$. We see, in the current section, that a limit can be equal to $f(x) = \pm\infty$, preserving continuity at the signed infinities. But the function $f(x) = \Phi$ is continuous even though its limit does not exist. Thus, limits do not provide an exhaustive test of continuity. This issue is taken up in the next section.

VI. Discussion

The transreal numbers provide a total arithmetic which brings many practical advantages for the designer of computer processors and programming languages. They might also bring advantages to the physical sciences.

As is well known, modern floating-point standards, such as [3], are total systems in that they provide infinities and an object NaN that allow the standard floating-point operations to be applied to any floating-point representation with the result being an object in floating-point representation. The object NaN is defined to be an exception so its occurrence must be defined in all extensions of the standards, such as in the mathematical libraries supplied with programming languages, and in user programs. By contrast, nullity is defined to be the fraction $\Phi = 0/0$. Its occurrence is defined by the axioms of transreal arithmetic [5] and by developments of it, such as the topology given in the present paper. It *does not* require a standards committee to meet to decide when a function should return NaN. Indeed, the lone programmer can decide this matter for himself or herself, simply by calculating the result. This is a far more satisfactory state of affairs.

Floating-point arithmetic is widely used because it deals with fractional numbers and is total. All of these advantages can be had by extending fixed-point arithmetic to a transfixed arithmetic [4] that models the transreal numbers, saturated at the infinities. As usual, this brings a huge benefit in massively reducing the circuitry needed to implement arithmetic, which means that more processors can be implemented on a chip. However, where the dynamic range of floating-point numbers is needed, this strategy is not appropriate. Floating-point arithmetic retains the advantages of relative error and huge dynamic range.

The design of a transreal processor simplifies the task of formally proving that the functional states of the processor are correct. Rather than prove theorems about the processor in a theorem-proving language, which requires the skills of a professional mathematician, it is sufficient to generate and test all of the states in a small version of the processor transarithmetically and then to scale the design up to a practical size, as outlined next.

It is known that there are at least six, distinct, transreal numbers ($-\infty$, -1, 0, 1, ∞, Φ) in any system that obeys the transreal axioms [5]. These six states, and two general states, can be encoded in three bits giving $2^3 = 8$ states, corresponding, say, to the numbers $-\infty$, -2, -1, 0, 1, 2, ∞, Φ in a transinteger processor. (Compare with the Appendix.) Now, it is common practice to design processors using a three-address-code [20]. This gives $(2^3)^3 = 512$ states per instruction with three bits per address. If the design is truly orthogonal, so that the instructions are independent of each other, then an instruction set with k instructions gives rise to $512k$ states. A typical Reduced Instruction Set Computer (RISC) has of order $k = 32$ instructions (some with further condition flags), giving $512 \times 32 = 16384$ states in all (but with the need to model the conditions). This number is small, subject to the need to model the conditions, so all of the states of a transreal processor can be generated, very quickly, by counting from 0 to 16383 on a conventional computer, leaving plenty of time to test that each of the states is valid. This test is purely transarithmetical, because the processor operates with transreal arithmetic and transreal arithmetic is total so there are no arithmetical error states nor any condition outside of transreal arithmetic which needs to be tested. Even where more complex arguments, such as floating-point numbers, are wanted, these can be modelled at a smallest size which can still be counted through in practical time. Thus, a smallest design can be tested exhaustively in practical time using a transarithmetical generate and test strategy.

It is common practice to design hardware parametrically so that, for example, the word size is a parameter. The task of generating a practical design is then simply to change this parameter to a practical size. In many cases it is possible to increase the word size by a small amount, say to four bits, and to check, by counting, that this design is also functionally correct. This provides a practical check that the design scales parametrically. More generally, one proves, by algebraic induction, that the design scales parametrically. This makes the process of formal proof much simpler than it would otherwise be. Indeed, it makes the proof strategy of generate and test so simple that it can be incorporated into existing Electronic Computer Aided Design (ECAD) tools, thereby making formal proof accessible to an electronic engineer without the support of a professional mathematician.

The electronic engineer needs only a knowledge of transreal arithmetic and its topology as set out here, and as exercised more fully in earlier papers, but the software engineer requires a knowledge of how transreal arithmetic applies in wider mathematical contexts so that he or she can implement total software that is exempt from arithmetical exceptions. This raises the question of how much of mathematics can be generalised by replacing real numbers with transreal numbers.

It has already been established, by formal proof, that transreal arithmetic is self-consistent and contains real arithmetic as a proper subset [5]. Hence, we may have considerable confidence that any arithmetical procedure can be generalised by substituting transreal numbers for real ones. The details of the generalisation need to be worked out, but, at least, we know that the generalisation, properly expressed, will not contradict any aspect of real arithmetic. But how far up the hierarchy of mathematics can these generalisations be made? We have just demonstrated (informally) that there are transreal topological and metric spaces which contain their real counterparts as a proper subset. Again, the details of a specific generalisation need to be worked out, but we can have some confidence that, properly expressed, no contradiction will arise with the standard topologies. We may reasonably hope that all of the mathematics in the hierarchy between arithmetic and topology generalises. This includes a very great part of mathematics and is all that is needed for many software applications.

An example may help to make clear why one needs to consider the details of a specific generalisation. We are interested in the question, is $f(x) = \Phi$ continuous? Now, nullity is unordered so there is nothing in real analysis or metric spaces which bears on this question. But if we proceed to a higher level of abstraction, the standard definitions of topological spaces tell us, amongst other things, that $f(x) = \Phi$ is continuous where $f: R^T \to \{\Phi\}$. This is certainly not as general as knowing that $f(x) = \Phi$ is continuous where $f: R^T \to R^T$, but, at least, we have been able to use the standard definitions of mathematics to get an answer to the question. It is now up to us how to generalise this result at the less abstract levels of metric spaces and real analysis. In this paper we have taken the extremely conservative step of making no change whatsoever at these lower levels, but of noting, by fiat at these lower levels, that $f(x) = \Phi$ is continuous. One might choose to examine the question of continuity in more detail, pursuing a mathematical aesthetic, but that is not our business here. We are concerned only to establish the elementary topology of the transreal numbers and to communicate this, in a timely manner, to practitioners of Computer Science so that they can use it in the practical design of processors and, later, of software.

But there is another motive for examining the transreal numbers. It is conceivable that they describe physical processes. Feynman [21] describes his work on classical and quantum electrodynamics and expresses the desire to develop a new physics that, amongst other things, resolves the problem of an infinite self-energy in the interaction between a moving electron and the electric field. One strategy he considered was to exempt an electron from acting on itself. In the end, he abandoned this strategy; but how might this self-exemption arise numerically? Feynman uses the Dirac Delta to describe the asymptotically brief interaction of the electric field with a distant electron, but how brief is the interaction of an electron with itself? Can it be so slow as to be asymptotically brief, or must it be of zero duration? If it is of zero duration then the box function takes on area nullity and the integral of this over the whole field (or a part of it) is zero, see [6], so that the electron has a zero self-interaction. We are not in a position to say whether this is physically realistic, all we can say is that it gives Feynman his self-interaction exemption as a transarithmetical property. In other words, if electrons and electric fields operate according to transreal arithmetic, not real arithmetic, then Feynman need look no further for a self-exemption; and we gain a demonstration of the physical reality of transreal numbers, that is, a demonstration that transreal numbers describe the behaviour of a physical system where real numbers do not.

There are many motives for exploring the transreal numbers: for their practical utility in the design of computer processors and software; for their mathematical aesthetic; and, perhaps, for their applications in physics. Whatever the reader's interests, we hope this exposition of the elementary, topological properties of the transreal numbers has thrown a little light on what makes transreal numbers interesting.

VII. Conclusion

The topology of the transreal numbers contains the topology of the real numbers as a proper subset and has the theoretical advantage that it removes all undefined and indefinite values from real analysis (calculus). It also removes the weird number from two's complement arithmetic, thereby removing a dangerous bug from practical programs in which the computed absolute value of a number can be negative.

VIII. Appendix

The table below shows how a three-bit, two's complement code is mapped onto the transdecimal numbers, that is the decimal numbers augmented with strictly transreal numbers, $-\infty, \infty, \Phi$.

Table. 7 Three-bit example of two's complement encoding

Decimal	2s Comp.	2s Comp.	Decimal
0	000		
1	001	111	-1
2	010	110	-2
∞	011	101	$-\infty$
		100	Φ

The figure below shows the two's complement codes laid out lexically in a clockwise direction from zero. The extended real-number line runs from $-\infty$ to ∞ , as usual, and nullity lies outside the extended real-number line. Thus, the topology of the transreal numbers is preserved in this encoding and the weird number with $abs(n) < 0$ is removed.

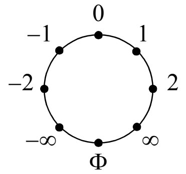

Fig. 3: Three-bit example of the two's complement encoding with codes laid out clockwise from zero in lexical order

IX. Acknowledgment

I would like to thank Oswaldo Cadenas for inventing the two's complement encoding of the transintegers using arbitrary codes for the strictly transreal numbers [4]. I remain responsible for the assignment of codes in lexical order [4], shown above, that preserves the piecewise topology of the transreal numbers.

X. References

[1] J.A.D.W. Anderson, "Perspex Machine VII: The Universal Perspex Machine" in *Vision Geometry XIV* Longin Jan Latecki, David M. Mount, Angela Y. Wu, Editors, Proceedings of the SPIE Vol. 6066 (2006).

[2] J-L Baer, *Computer Systems Architecture,* Pitman 1980.

[3] IEEE Standard 754 for Binary Floating-Point Arithmetic (ANSI/ IEEE Std 754-1985).

[4] O. Cadenas, *Two's Complement Transreal Coding,* UK Patent Application 0625735.6 for Reading University (22 Dec. 2006).

[5] J. A. D. W. Anderson, Norbert Völker, Andrew A. Adams "Perspex Machine VIII: Axioms of Transreal Arithmetic" in *Vision Geometry XV,* Longin Jan Lateki, David M. Mount, Angela Y. Wu, Editors, Proceedings of SPIE Vol. 6499 (2007).

[6] J.A.D.W. Anderson, "Perspex Machine IX: Transreal Analysis" in *Vision Geometry XV* Longin Jan Latecki, David M. Mount, Angela Y. Wu, Editors, Proceedings of the SPIE, Vol. 6499 (2007).

[7] B. Mendleson, *Introduction to Topology,* Blackie & Son, London, (1963).

[8] M.L. Boas, *Mathematical Methods in the Physical Sciences,* Wiley, (1966).

[9] S. Swierczkowski, *Sets and Numbers,* Routledge & Kegan Paul, (1972).

[10] L. Arkeryd "The Evolution of Nonstandard Analysis" *The Mathematical Association of America Monthly,* no.112, pp. 926-928, (Dec. 2005).

[11] G. Leibman, "A Nonstandard Proof of the Fundamental Theorem of Algebra" *The Mathematical Association of America Monthly,* no.112, pp. 705-712, (Dec. 2005).

[12] J.H. Conway, *On Numbers and Games,* 2nd Edition, A.K. Peters Ltd (2001).

[13] W.A. Sutherland, *Introduction to Metric and Topological Spaces,* Clarendon Press, Oxford (1975).

[14] M. Spivak, *Calculus,* W.A. Benjamin, London, (1967).

[15] M.A. Armstrong, *Basic Topology,* McGraw-Hill, London, (1979).

[16] M.D. Crossley, *Essential Topology,* Springer-Verlag, London, (2005).

[17] J. Dixmier, *General Topology,* Springer-Verlag, New York, (1984).

[18] E.M. Patterson, *Topology,* Oliver and Boyd, Edinburgh, (1956).

[19] I.M. James, *History of Topology,* North-Holland, (1999).

[20] J.P. Bennett, *Introduction to Compiling Techniques,* 2nd edn., McGraw-Hill (1996).

[21] R.P. Feynman, "The Development of the Space-Time View of Quantum Electrodynamics" in *Science,* New Series, vol. 153, no. 3737, pp. 699-708 (Aug. 12, 1966).