**PERSPEX MACHINE XII:
TRANSFLOATING-POINT AND TRANSCOMPLEX ARITHMETIC
WITH APPLICATIONS IN MATHEMATICAL PHYSICS**

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Perspex Machine XII: Transfloating-point and transcomplex arithmetic with applications in mathematical physics

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Transreal arithmetic received considerable public comment. We briefly review its development, set it in its proper mathematical context, and give a tutorial which is accessible to the general reader, including secondary-school children. Building on this introduction, we show that transfloating-point arithmetic is more efficient and safer than IEEE floating-point arithmetic. Then we define transcomplex numbers as three-tuples of a transreal radius, cosine, and sine. We show that a careful arrangement of the ordinary algorithms of complex arithmetic holds for all transcomplex numbers, including those with zero and non-finite components. An implementation of transreal and transcomplex arithmetic is given as an online appendix. We apply transarithmetic in several settings throughout the paper. In particular, we use it to compute gravitational and electrostatic forces at and near a singularity.

Keywords: transarithmetic, space-time singularity, NaN, floating-point arithmetic, Riemann sphere.

AMS Subject Classification (2010): 03H15; 30B60; 70A99; 83C10; 83C75; 97F99; 11U10

C.R. Category: G.1.0 Computer Arithmetic

1. Foreword

This paper presents what ought to be a simple mathematical proposition that just as division by zero develops *transreal* arithmetic as a proper superset of real arithmetic [1] so division by zero develops *transcomplex* arithmetic as a proper superset of complex arithmetic. But such is the psychological resistance to division by zero – amongst general readers, scientists, and reviewers – that we must make special efforts to present this work in a palatable way. Accordingly, we present a longish introduction in which alternative ways of dealing with division by zero are discussed. It is shown how transreal arithmetic differs from each of

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these and how, in some cases, it has properties analogous to earlier methods for dealing with division by zero. For example, every transreal number is a *definite number,* but, by analogy with *domain theory* [2], the *finite numbers* can be used to model *quantities* whose magnitude and sign are known exactly; *positive infinity* and *negative infinity* can be used to model quantities whose sign is known, but whose magnitude is known only to be large; and *nullity* can be used to model quantities where nothing is known about the sign or magnitude of the quantity. It is a grievous, but common, error to mistake such a *model of quantities* for the *properties of numbers* that are used to construct the model. Later in the paper, we present a model of finite and non-finite forces whose resultant is computed as a combination of both topological and arithmetical considerations. Just as in ordinary physics, the properties of numbers alone are not sufficient to compute a resultant force. We model physical forces as obeying some mathematical model, such as that implied by a parallelogram rule for *vector addition,* in which the addition of physical forces is modelled by the mathematical addition of vectors, which, in turn, is implemented in terms of arithmetic on numbers. Thus, in the addition of infinite and nullity forces, we shall arrive at an example of the fact that transreal arithmetic is a definite arithmetic which can be used to model quantities that are ordinarily taken to be definite, indefinite, or undefined. But we obtain more. Transreal arithmetic makes the sums of all forces definite. This permits us to compute definite properties, whether finite or non-finite, at physical singularities.

Transreal arithmetic is *total,* which means that every defined operation of the arithmetic can be applied to any transreal number(s) with the result being a transreal number. Hence, each step of a computation can be completed, even if a sequence of steps does not converge to a solution or terminate. Thus, *totality* has no direct bearing on *computability,* other than to remove some error states which complicate the analysis of a program implemented in a *partial* arithmetic. Ordinary real and complex arithmetics are partial because division by zero is not supported.

In general, a transreal algorithm lays out a finite number of *computational paths* that identify all of the elements of a *solution set* which classify all of the finite, infinite, and nullity solutions. The algorithm then evaluates each of these paths and, if a best answer is wanted, selects the one with the highest *information content.* In many problems, finite numbers convey more information than infinite numbers which, in turn, convey more information than nullity. But this is not a universal rule; it is possible to construct abstract situations in which any particular transreal number has the highest information content. The issue is that transreal arithmetic is total so it can carry out computations at every step of a computational path, but it is up to the user of the arithmetic to chose which path is wanted. That choice may be formalised in an algorithm or may be left entirely to the user.

After the introduction, we present a tutorial on transreal arithmetic which is so simple that it has been used, successfully, with secondary-school children. It should not present the reader with any difficulty; but the reader must practice the arithmetic sufficiently to internalise it, otherwise it will not be possible to follow the proofs given later in the paper. In our experience, some reviewers are prone to a fault in this regard. Hasty reviewers sometimes develop an extension of transarithmetic, prove that it leads to a contradiction, and reject the paper, citing this contradiction. In every case, to date, the reviewers have succeed only in proving that their own extensions of transarithmetic are faulty – a situation which might be avoided by an exchange of letters with the author, via a journal's editor.

We then make IEEE floating-point arithmetic [3] [4] more efficient by replacing the *minus-zero* state with the nullity state, by moving the positive and negative infinity states to the most extreme positive and negative bit patterns, and by replacing all of the NaN states with real numbers. This ensures that every binary state encodes a unique number. This almost doubles the arithmetical range of real numbers described by the floating-point bits. We show that IEEE ordering is far more complicated than transreal ordering and argue that this makes it difficult for programmers to use IEEE arithmetic correctly. In other words, we hold that IEEE floating-point arithmetic is a risk to the proper operation of both mundane and safety-critical systems.

We then define that a transcomplex number is a three-tuple, (r, c, s), of a transreal radius, r, being the *Euclidean transmetric* [5] of a vector in our *extended-complex plane* or on the *axle at angle nullity,* where c and s are, respectively, the *transreal sine* and *transreal cosine* [6] this vector makes with the positive transreal axis. The *ordinary extended-complex plane* is the complex plane augmented with a single point at infinity, whereas our extended-complex plane is further extended by a circle of points at infinite radius and finite angle. Thus, our extended-complex plane is a proper superset of the ordinary extended-complex plane. We then define the operations of addition, subtraction, multiplication, and division on these three-tuples. Various ordinary forms of complex numbers are bijective with a proper subset of the three-tuples, as can be seen immediately. With the usual notation, [7] [8], *Cartesian-complex* numbers, (x, y), are given by (rc, rs) with x on the real axis and y on the imaginary axis; *polar-complex* numbers, (r, θ), in the principal range, $-\pi < \theta \le \pi$, are given by $(r, \text{arctan2}(c, s))$; *Eulerian-complex* numbers, $n = re^{i\theta}$, in the same principal range, are given by $n = (rc, irs)$, where i is the complex unit, $(1, 0, 1)$, in three-tuple form; and *Riemannian-complex* numbers, being all of the Cartesian-complex or polar-complex or Eulerian-complex numbers in the complex plane, augmented with *complex infinity,* arise from projection of the Riemann sphere, [8] [9] [10], and are described by the corresponding transcomplex numbers, with the point at infinite radius and nullity angle, (∞, Φ, Φ), replacing the ordinary complex infinity. We prove that where the operations of addition, subtraction, multiplication, and division are defined on these ordinary forms, they produce the same result as mapping the ordinary form to three-tuples, performing three-tuple arithmetic, and mapping the result back to the ordinary form. In this sense, transcomplex arithmetic is a universal complex arithmetic which contains all of the ordinary complex arithmetics as proper subsets. We then present preliminary results on the transcomplex exponential and transcomplex logarithm, and use these to define the transcomplex operation of raising an arbitrary transcomplex number to the power of an arbitrary transcomplex number. In particular, we can evaluate 0^p and $-\infty^p$ as definite *transnumbers* for any transcomplex power, p, including zero.

We show that transcomplex arithmetic supports a superset of the *Riemann sphere* which is extended by the inclusion of an axle and a circle. The axle passes through the 'north' pole of the sphere and terminates on the centre of the sphere. This axle projects onto the *axle at angle nullity.* The circle is centred on the north pole and lies in a plane parallel to the complex plane. This circle, with unit diameter, projects onto the *circle at infinity* or *rim at infinity.* The axle has two isolated points on it: the north pole and the southern pole-star. The pole star lies at unit distance below the origin and projects onto the *point at nullity* [11]. The axle supports unsigned arithmetic, while signed arithmetic is supported by the circle, together with the sphere, but excluding its north pole. Ordinarily, the north pole of the Riemann sphere maps to a special, non numerical, object called *complex infinity,* which models all points at infinite radius, regardless of angle. In our model the north pole maps to (∞, Φ, Φ) which is the unique point at infinite radius and non-finite angle. All of the points (∞, c, s), with real c and s such that $c^2 + s^2 = 1$, are points at infinite radius and finite angle, which make up the circle at infinity. Hence, complex analysis may be extended to model functions whose modulus grows monotonically with no real bound, and whose angle either does, (∞, c, s), or does not, (∞, Φ, Φ), converge to a finite angle. This introduces new convergence results and makes transcomplex analysis a proper superset of transreal analysis. By contrast, ordinary complex analysis is not a superset of real analysis, because real analysis has infinite limits of and approaching one of exactly two signed infinities, but complex analysis has infinite limits of and approaching exactly one unsigned infinity. It would be interesting to know if any of the new convergence results, or the geometrical unification of signed and unsigned arithmetics, are useful in mathematical physics.

In the tutorial we show how to use transarithmetic to evaluate equations in Newtonian physics. We have examined Newton in translation [12] [13] and have checked diagrams in an original [14]. Analogously to Newton's first law of motion, we say that, "Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by non-zero and non-nullity forces impressed." Compare with [12] p. 416. Analogously to the second law, we say that, "A change in motion is given by any or all of the satisfied equations $F = ma$, $m = F/a$, $a = F/m$ and takes place along the straight line in which a force is impressed." Here the equations use transreal or transcomplex variables in an elementary notation and are readily extended to Newton's, or modern, differential form. This statement is a very considerable departure from Newton's original law, [12] p. 416, but is a lesser departure from the corpus of physics Newton presents [12] [15]. It is notable that Newton does much of his work in *proportions* (unsigned ratios). This follows the ancient Greek practice of representing quantities by proportions. It also has the advantage, for us, that ratios immediately generalise to *transnumbers* so that Newton's statement of physics can be generalised so that it applies to infinite and nullity quantities. Analogously to the third law, we say, "To any action, F, there is always an opposite and equal reaction, $-F$; in other words, the actions of two bodies upon each other are always equal and always opposite in direction." Compare with [12] p. 417. The variable, F, may be transreal or transcomplex. We shall also find that Newton's statement of the parallelogram law applies to infinite forces, though modern statements of the law do not.

The reader, who is of a practical bent of mind, will want results of the above sort, where Newtonian physics is re-cast in transmathematics so that familiar physical equations apply at singularities. But the reader who is interested in the history of science might appreciate our demonstration that all of the mathematics in Newton's *Philosophiae Naturalis Principia Mathematica* is extended by transmathematics so that it would be possible to re-write that work so that all of its results apply at singularities. Transmathematics also extends the methods of proportions given in Euclid's *Elements.* As transmathematics extends historical works in mathematics, it would seem reasonable to suppose that it might have an influence on the future development of mathematics; and even its mundane results, such as making floating-point arithmetic more efficient and safer, might influence applications of general-purpose digital computing.

2. Introduction

The Perspex machine [16] was introduced [17] as a theoretical computer that carries out all computations geometrically, using transreal co-ordinates. The transreal number *nullity,* Φ, was defined [11] to be the unique number which is zero divided by zero. Thus, $\Phi = 0/0$. It was used to solve certain degenerate geometrical problems [11]. Later, several transreal arithmetics were developed, starting with [18]. In each case, a set of transreal numbers was defined, then a careful selection of the established algorithms of real arithmetic was made. The objective was to retain real arithmetic and to obtain useful properties for the *strictly transreal* numbers that are not real. Notions of usefulness varied, but always dealt with digital computation and, later, with mathematical physics. Each arithmetic was formalised by giving its multiplication tables. This methodology sets the development of transreal arithmetic apart from the development of number systems in modern mathematics. Firstly, the methods used are algorithmic, not axiomatic, and, secondly, the overriding objective is usefulness, not any aesthetic of pure mathematics. This fundamental development, of transreal arithmetic, is a product of applied mathematics, specifically of computer science, not of pure mathematics. Thus, the development of transreal arithmetic reprises the history of mathematics, to the extent that arithmetic was used to solve practical problems before it was axiomatised. Indeed, transreal arithmetic provides an alternative history: because it uses only the algorithms of real arithmetic, it might have been obtained in ancient times, and might have lead to an earlier axiomatisation of an arithmetic that allows division by zero. This narrative raises psychological difficulties for some people who are reluctant to accept that the arithmetical methods they already know, do allow division by zero. In our experience, children of 12 to 16 years of age have the necessary mathematical preparation to understand transreal arithmetic and have no psychological inhibitions to their understanding. Our tutorial on transreal arithmetic is aimed at this group of students, though students at the younger end of the range will generally require the assistance of an instructor to explain the text to them. A preferred form of transreal arithmetic was arrived at in [16] which uses two signed infinities: $\infty = 1/0 \equiv k/0$ and $-\infty = -1/0 \equiv -k/0$, where $k > 0$. This was axiomatised in [1], where a machine proof of consistency was given. This proof shows that transreal arithmetic is consistent if real arithmetic is. But, at the time of writing, no proof of the consistency of real arithmetic is known. After this paper was accepted for publication, and a pre-print was published on the World Wide Web, the BBC reported the development of transreal arithmetic. These reports attracted considerable public comment.

Subsequently, the topology of transreal numbers was developed [5]. It was recognised that nullity and the infinities are *non-finite numbers,* in contrast to the real numbers, each of which is a *finite number*. The methodology for developing transreal arithmetic was then used to develop transcomplex arithmetics. In each case, a set of transcomplex numbers was defined, then a careful selection of the established algorithms of complex arithmetic was made which applied to all of the transcomplex numbers. Later, a geometrical construction was found which justified this choice. In the current paper, transcomplex numbers are defined as three tuples, (r, c, s), where the radius, r, is the Euclidean transmetric [5] of a transcomplex vector; and c and s are, respectively, the transreal cosines and sines [6] this transcomplex vector makes with the transreal x-axis. Transcomplex arithmetic unifies and contains: the whole of Cartesian-complex arithmetic, both polar-complex and Eulerian-complex arithmetics in the principle range of angles, arithmetic on Riemannian-complex numbers, and both transreal and real arithmetics. In this sense it is a universal complex arithmetic which can stand in for any of the established complex arithmetics. But, just like transreal arithmetic, this raises psychological difficulties for some people who are reluctant to accept that the methods of complex arithmetic they already know, do allow division by zero and do, therefore, operate at singularities. These psychological difficulties are symptomatic of the fact that the transarithmetics result from the acceptance of a new number, nullity, and a paradigm shift in the application of the accepted methods of arithmetic.

In the week following the BBC broadcasts, there were 40 000 downloads of the axioms paper [1]. Within three weeks, Google reported 100 000 web pages relating to transreal arithmetic, and the arithmetic was reported by journalists from several nations. Within this period, 1 000 people corresponded with the author, of whom 100 asked technical questions. These questions were answered individually, in the published version of [1], and in [5]. The present paper gives answers to the remaining technical questions so that the general public now has a detailed, but not necessarily definitive, answer to all of the questions asked.

It is unusual to promote the public understanding of science in a technical paper, but such is the level of interest in the topic of division by zero that it seems reasonable to use this introduction to deal with popular questions, in addition to technical ones. When dealing with any question we employ the *Principle of Charity* [19]. That is, we make the best argument we can for the questioner and explore the improved question so as to obtain as much information as possible about the matter in hand. Sometimes this leads to a clearer understanding of each party's position and, sometimes, it leads to a scientific advance. Such is the case here, where a thorough answer to a question about floating-point arithmetic, results in a more efficient and safer interpretation of floating-point bits. We shall come to that presently, but we begin with a more fundamental question.

It has been asked if *nullity* is identical to *undefined.* In order to give a meaningful answer to this question, we propose various definitions of *undefined* and then answer the question in these terms. This might satisfy the questioner or it might be that the questioner had in mind a different definition than the ones we consider here or, after reading our answers, the questioner might arrive at new questions, in which case a discussion between us will be needed to advance our mutual understanding. Let us now turn to the question posed.

If *undefined* is the same as *not being defined* then the answer, to this question, is straight forward: nullity has a definition, it is defined to be the unique number which is zero divided by zero, so it is not undefined. This is not a trite answer, we can use the definition of nullity, together with the operations of transreal arithmetic, to obtain arithmetical solutions where ordinary arithmetic cannot. For example, the following are undefined formulae in real arithmetic: $0/0$, ∞/∞, $\infty-\infty$, 0^0, $\log 0$. But, by definition, we have $\Phi = 0/0$ and, using the methods of transreal or transcomplex arithmetic, presented in this paper, we compute the following identity: $\Phi = \infty/\infty = \infty-\infty = 0^0$. But we also compute $\exp(-\infty) = 0$, whence $\log 0 = -\infty$. In other words, some undefined formulae evaluate to nullity and some evaluate to an infinity. Therefore, it is impossible that nullity is equal to all undefined formulae. (All of the numerical examples in this journal paper are evaluated in the software package that is included as an on-line appendix. The examples just given appear as *test_1* in the on-line appendix and are evaluated, by hand, later in the paper.)

If *undefined* is the same as *not being able to begin a computation* then we have already answered the question. The above formulae cannot be computed in real arithmetic. In some cases, the formulae cannot even be stated in real arithmetic, because they involve the infinity symbol, ∞, which is undefined in real arithmetic. These formulae are ∞/∞ and $\infty-\infty$. In two cases, the formulae involve only real numbers, but are still undefined. These formulae are $0/0$ and 0^0. In the latter case, we can try to evaluate the formula: $0^0 = e^{\log 0^0} = e^{0\log 0}$. But the term 0^0 in $\log 0^0$ is undefined, and $\log 0$ is also undefined. Thus, when using real arithmetic, we can neither begin the computation nor take any steps in evaluating it. However, we can evaluate all of these formulae in transarithmetic. As stated above, we obtain $\Phi = \infty/\infty = \infty-\infty = 0^0$ and $\log 0 = -\infty$ so, again, some undefined formulae evaluate to nullity and some evaluate to an infinity. Therefore, it is impossible that nullity is equal to all formulae that are undefined.

For the reader's benefit, we evaluate all of the above formulae later in the paper. See Equations (5), (10). But let us now pursue the question of computability on the assumption that the initial formulae can be stated in a way that allows a computation to begin.

If *undefined* is the same as *incomputable* then we turn to Kleene for an analysis of the logic of computability. Kleene develops a logic of undefined computations in his treatise on metamathematics [20]. In this text, Kleene employs the concept of an *object language* that is analysed from the point of view of an *observer language.* He also gives a definition, [20] ch XII, of *undefined.* This definition is the strongest possible that contains the whole of Boolean logic – the fundamental logic of truthfulness and falsehood that is used in establishing the foundations of mathematics and computer science. Kleene states his definitions in terms of Turing machines and their equivalents, but his argument holds for some stronger machines [21] and all weaker ones. In this latter case, it holds for all of today's digital electronic computers. In Kleene's logic: *true* (T) means that a computer has determined that a sentence has the truth value T; *false* (F) means that a computer has determined that a sentence has the truth value F; and *undefined* (U) means that a computer has not yet determined whether the truth value of a sentence is T or else F – indeed, it will never do so if the sentence is incomputable in the current model of computation, here Turing computation and its equivalents.

Table 1: Kleene's truth table for the logical disjunction, *or.*

or	T	F	U
T	T	T	T
F	T	F	U
U	T	U	U

We illustrate Kleene's argument with his truth table, [20] page 334, for the logical disjunction, *or,* as shown in Table 1, above. Consider the sentence *"a or b."* If either or both of *a, b* are true then the sentence is true. This is shown in the first row and column in the body of the table. In particular, notice that the element in the first row and third column, in the body of the table, says that the sentence is true when *a* is true and *b* is undefined. As Kleene argues, all that is needed for the disjunction to be true is that one of *a, b* is true – it is irrelevant whether the other is true, false, or undefined. The element in the second row and second column, in the body of the table, says that the sentence is false if both *a, b* are false. The remaining entries, in the body of the table, are all undefined, because there is no evidence available to the computer that any of *a, b* is true. Kleene states that this is the strongest possible interpretation of undefined that contains the whole of Boolean logic. The Boolean terms are shown in the top-left, two by two, block of elements in the body of the table. Kleene's definition of undefined can be weakened by changing any T, in the body of the table, to an F or a U, and by changing any F, in the body of the table, to a U. But if any element in the top-left, two by two, block is changed then the definition of undefined does not contain the whole of Boolean logic.

Returning to the question: if undefined in the object language of transreal arithmetic, means that U is identical to Φ, then the question is vacuous. Nullity, is not a truth value. It is the wrong type of mathematical object to appear in a truth table. But suppose that we adopt Kleene's logic in an observer language and ask if Φ, in the observed language, behaves such that *"a op b"* is identical to Kleene's table when *op* is some, particular, binary operation of transreal arithmetic that is substituted for *or*, and where Φ is substituted for U. On consulting [1] we find that all transreal operations involving Φ produce Φ as a result. Therefore, it is impossible for us to find any sentences of the form *"*Φ *op* $b \to c$*"* or *"a op* $\Phi \to c$*,"* where $c \neq \Phi$, so we cannot match the entries "U or T → T" or "T or U → T" in Kleene's table. Consequently, nullity is not identical to Kleene's notion of undefined in the language of transreal arithmetic, nor in any meta language obtained by a simple substitution of terms. But we could use a complicated substitution of terms. We could Gödelise the axioms of transreal arithmetic, encode all of Kleene's tables with an arbitrary substitution of terms, and supply a machine to evaluate sentences in the Gödelised arithmetic. But this is completely arbitrary. We could just as well encode 42 as undefined. And we would have to add a machine to the transarithmetics, thereby creating yet another formal system. (We do not want to add an arbitrary machine to the transarithmetics, because we are already committed to adding the Perspex machine to these arithmetics). We summarise this by saying that there is no non-arbitrary way to make Kleene's notion of undefined identical to transreal nullity.

If we weaken Kleene's notion of undefined, in any of the ways we have identified, then we end up with a logic that does not contain the whole of Boolean logic or which forbids one or both of the simplifications "U or T → T," "T or U → T." But Boolean logic is used universally in digital computers. And, in almost all computer languages, the simplifications "U or T → T," "T or U → T" are used to terminate evaluation of a logical sentence, without evaluating all of the terms in the sentence, as soon as the sentence's truth value is known. So, if we take nullity as identical to a weak notion of undefined then we must give up Boolean logic or modify the most common strategy for evaluating logical sentences in a computer. We are free to do this, providing we are willing to live by the consequences of doing it. We acknowledge, therefore, that nullity could have all the properties of undefined if we are willing to accept a departure from the ordinary practice of computation. But note, carefully, that we rebut the suggestion that nullity can be taken identical to undefined, because nullity has some additional properties that none of the definitions of undefined has. We present two such properties later, thereby refuting the suggestion that nullity is undefined.

If *undefined* is the same as *bottom type* then we delve deeper into the recent history of mathematics. Kleene's work on computability was preceded by Russell's work on type theory. Russell developed a relative hierarchy of types with the freedom to choose the first type arbitrarily. Whitehead and Russell used this theory in the development of mathematics from its logical foundations [22]. Today, type hierarchies are used in the design of computer languages [23]. In modern type theory, there is a bottom type which is a subtype of every other type. For example, the bottom type is a subtype of the type *number* and of every other type, whether mathematical or not. In particular, it is a subtype of the type *goldfish riding a unicycle.* Now, nullity is the unique number which is zero divided by zero, written in transreal or transcomplex form, and in any other forms yet to be developed. Nullity is of the type number, but it is not itself a type. In particular, nullity is not a subtype of *goldfish riding a unicycle.* Therefore, nullity is not identical to the bottom type. (And it stretches Charity to the extreme to allow that a technically competent person could believe otherwise.)

If *undefined* is the same as *bottom element* then we move closer to the present day in the history of mathematics. Kleene's work on computability was succeeded by Scott's work on denotational semantics [2]. This is an observer language that uses a bottom element which means that nothing is known about a computation. We are free to develop applications of denotational semantics in which the bottom element is as strong as, or weaker than, Kleene's undefined. If we use a bottom element as strong as Kleene's undefined, then the above analysis applies and we hold that nullity is distinct from the bottom element, but we acknowledge that nullity has all of the properties of the bottom element, if we are willing to accept a departure from the practice of ordinary computation. Now, let us examine applications of the bottom element that are weaker than Kleene's undefined. The bottom element can be used so that when it is combined with any element of the observer language of denotational semantics, it results in the bottom element. This is the same behaviour as when nullity is combined with any element in the observed language of transreal arithmetic. We could, therefore, find a translation of these transreal operations into denotational semantics. Hence, we are free to read nullity as the bottom element. For example, if we consider the set of extended-real numbers, made up of the real numbers augmented with the transreal infinities, then a result of nullity means that nothing is known about the extended-real result of a computation. But we are equally free to accept nullity as a known value and operate on the whole set of transreal numbers so that every terminating computation produces a known result. We summarise this position by saying that we are free to read nullity as having all of the properties of the weakest reading of the bottom element that we have considered, but we are equally free to read nullity as a known number. Whichever case we choose, nullity is distinct from the bottom element because it has properties that the bottom element does not have.

We now present two arithmetical properties which nullity possesses, but which are not possessed by any of the definitions of undefined that we have considered. Recall that nullity is defined to be the unique number that is zero divided by zero. Consequently, nullity has a numerator of zero and a denominator of zero. By contrast, the numerator of undefined is undefined and the denominator of undefined is undefined. But zero is not equal to undefined. Therefore, nullity and undefined are not identical. As a second example, consider the search for a number which is greater than zero and less than zero. There is no such number in real or transreal arithmetic, but on any of the above readings we say that this target number is undefined. If we now suppose that undefined is identical with nullity we get a contradiction. The axiom of quadrachotomy, axiom [A30] in [1], states, as two of its four cases, that nullity is not less than zero and is not greater than zero. Hence, nullity cannot be a number which is less than zero and greater than zero. Therefore, nullity and undefined are not identical. In conclusion, nullity is a number with numerical properties that are distinct from any definition of undefined that we have considered.

We summarise the whole of this argument by saying that nullity is a unique number and is not identical to undefined; nonetheless nullity can be used to model undefined in various circumstances. For example, in computer algebra or theorem proving software, we would be happy to describe undefined results by the empty set, but in the design of an electronic Perspex machine, we do not want the architectural overhead of describing undefined terms by a set. Instead, we describe these terms by the number nullity. For example, the transreal logarithm [3] has $\log\Phi = \Phi$ as a consequence of evaluating the power series $\exp\Phi = \Phi$ and, where the real logarithm, $\log x$, is undefined, on $-\infty < x < 0$, the transreal logarithm is defined to have $\log x = \Phi$ on $-\infty \leq x < 0$. This totalises the transreal logarithm so that it can be applied to any transreal x, with the result being a transreal number. We may then choose how to treat $\log x = \Phi$. We may choose to treat Φ as everywhere undefined so that $\log x = y$ is undefined whenever $y = \Phi$. Alternatively, we may classify the computational paths that lead up to the computation of any particular instantiation of $\log x = \Phi$ into two classes: first, those that lead to an exact result in the case $x = \Phi$ and, second, those that lead to an otherwise undefined result in the case $-\infty \leq x < 0$. We may then program an appropriate behaviour along each path. For example, we may use the transcomplex logarithm, presented later, to compute a unique result in every case $-\infty < x \le \infty$ or $x = \Phi$, though the transcomplex logarithm is itself totalised by $\log(-\infty) = \Phi$. This totalisation does not nullify the computation of $-\infty^p$ because there is another computational path, with more information, that computes a definite result in the circle at infinity. Thus, we can let computations run, without halting on an exception, unless we want to halt, even where the function is totalised by a boundary condition. This behaviour is reminiscent of the use of NaNs in IEEE floating-point arithmetic [3].

It has been asked if *nullity* is *NaN* as defined in the IEEE standard for floating-point arithmetic [3]. The answer is straight forward, nullity is a unique number and the NaNs are a class of many distinct objects that are, as the acronym says, *N*ot *a* *N*umber ([3], p. 8). Thus, nullity differs both quantitatively and qualitatively from the NaNs. But, to be Charitable, we now examine the differences and, later, come to the conclusion that floating-point arithmetic is more efficient and safer when it uses the unique number nullity than when it uses the class of NaNs.

We tabulate, below, a comparison between transreal arithmetic and IEEE floating-point arithmetic. The left part of the table, headed "Transreal arithmetic," relates to transreal arithmetic as axiomatised in [1]. The right part of the table, headed "IEEE floating-point arithmetic," relates to floating-point arithmetic as specified in [3]. Note that this standard is ambiguous so the entries in our table are open to question. We cite the parts of the standard that support our reading. Both the left and the right parts of the table are subdivided into two columns. The first column, headed "Return value," shows the value that an arithmetical operator returns. The second column, headed "Equality," describes the behaviour of the, optional ([3] §5.7), IEEE equality operator. Specifically, $a = b$ means that the equality operator, =, returns *true,* and $a \ne b$ means that the equality operator, =, returns *false.* The equality and relational operators are discussed later. Within the table, op, is one of binary addition, subtraction, multiplication, or division; and x is an arbitrary transreal or floating-point number, as the case may be. The square root operator is defined specially in the standard ([3] § 5.2) and is shown in the table. Finding the integer remainder is also defined in the standard ([3] § 5.1), but this operation has not been developed in transreal arithmetic so it is not shown in the table. Notice that transreal arithmetic agrees with IEEE floating-point arithmetic in the first two rows of the table, but disagrees in the remaining twelve rows. In particular, every operation involving nullity behaves differently from the corresponding operation on a NaN.

Transreal arithmetic has a single zero, which is isomorphic with the zero in real, Cartesian-complex, Eulerian-complex, Riemannian-complex, and transcomplex arithmetic; but IEEE floating-point arithmetic has two zeros: zero, 0, and minus zero, -0 ([3] § 3, 3.2, 5.7). The IEEE zeros are equal, but not identical, and sometimes behave differently. Zero sums are not shown in the table because their sign depends on the rounding mode in which IEEE arithmetic is performed ([3] § 4). Transreal arithmetic has a single nullity; but IEEE floating-point arithmetic has at least two NaNs: a signalling NaN and a quiet NaN ([3] § 3, 6.2). Quiet NaNs should carry diagnostic information so a resultant NaN can sometimes be different from an argument NaN ([3] § 6.2). Transreal arithmetic is reflexive, which is to say that $x = x$ for all numbers x. Similarly, real, Cartesian-complex, polar-complex, Eulerian-complex, Riemannian-complex, and transcomplex arithmetics are all reflexive; but IEEE floating-point arithmetic is nonreflexive, it has $\mathrm{NaN}_i \ne \mathrm{NaN}_i$ for all i ([3] § 5.7). In IEEE arithmetic, the signs of products and quotients are the exclusive *or* of the signs of their arguments ([3] § 6.3). This creates a variety of signed results so that the right-hand part of the table has many more entries than the left-hand part. This demonstrates that IEEE arithmetic is different from, and is far more complicated than, transreal arithmetic.

Table 2: Comparison of transreal and IEEE floating-point arithmetical operators.

Transreal arithmetic		IEEE floating-point arithmetic	
Return value	Equality	Return value	Equality
$-0 \to 0$	$-0 = 0$	$-0 \to -0$	$-0 = 0$
$-\infty \to -\infty$	$-\infty = -\infty$	$-\infty \to -\infty$	$-\infty = -\infty$
$-\Phi \to \Phi$	$-\Phi = \Phi$	$-\text{NaN}_i \to \text{NaN}_j$	$-\text{NaN}_i \neq \text{NaN}_j$
$x \text{ op } \Phi \to \Phi$ $\Phi \text{ op } x \to \Phi$	$x \text{ op } \Phi = \Phi$ $\Phi \text{ op } x = \Phi$	$x \text{ op } \text{NaN}_i \to \text{NaN}_j$ $\text{NaN}_i \text{ op } x \to \text{NaN}_j$	$x \text{ op } \text{NaN}_i \neq \text{NaN}_j$ $\text{NaN}_i \text{ op } x \neq \text{NaN}_j$
$\sqrt{0} \to 0$	$\sqrt{0} = 0$	$\sqrt{0} \to 0$	$\sqrt{0} = -0 = 0$
$\sqrt{-0} \to 0$	$\sqrt{-0} = 0$	$\sqrt{-0} \to -0$	$\sqrt{-0} = -0 = 0$
$0 \div 1 \to 0$	$0 \div 1 = 0$	$0 \div 1 \to 0$ $(-0) \div (-1) \to 0$	$0 \div 1 = -0 = 0$ $(-0) \div (-1) = -0 = 0$
$0 \div (-1) \to 0$	$0 \div (-1) = 0$	$0 \div (-1) \to -0$ $(-0) \div 1 \to -0$	$0 \div (-1) = -0 = 0$ $(-0) \div 1 = -0 = 0$
$1 \div 0 \to \infty$	$1 \div 0 = \infty$	$1 \div 0 \to \infty$ $(-1) \div (-0) \to \infty$	$1 \div 0 = \infty$ $(-1) \div (-0) = \infty$
$-1 \div 0 \to -\infty$	$-1 \div 0 = -\infty$	$-1 \div 0 \to -\infty$ $1 \div (-0) \to -\infty$	$-1 \div 0 = -\infty$ $1 \div (-0) = -\infty$
$0 \div 0 \to \Phi$	$0 \div 0 = \Phi$	$0 \div 0 \to \text{NaN}_i$ $(-0) \div 0 \to \text{NaN}_i$ $0 \div (-0) \to \text{NaN}_i$ $(-0) \div (-0) \to \text{NaN}_i$	$0 \div 0 \neq \text{NaN}_i$ $(-0) \div 0 \neq \text{NaN}_i$ $0 \div (-0) \neq \text{NaN}_i$ $(-0) \div (-0) \neq \text{NaN}_i$
$\infty \div \infty \to \Phi$ $(-\infty) \div \infty \to \Phi$ $\infty \div (-\infty) \to \Phi$ $(-\infty) \div (-\infty) \to \Phi$	$\infty \div \infty = \Phi$ $(-\infty) \div \infty = \Phi$ $\infty \div (-\infty) = \Phi$ $(-\infty) \div (-\infty) = \Phi$	$\infty \div \infty \to \text{NaN}_i$ $(-\infty) \div \infty \to \text{NaN}_i$ $\infty \div (-\infty) \to \text{NaN}_i$ $(-\infty) \div (-\infty) \to \text{NaN}_i$	$\infty \div \infty \neq \text{NaN}_i$ $(-\infty) \div \infty \neq \text{NaN}_i$ $\infty \div (-\infty) \neq \text{NaN}_i$ $(-\infty) \div (-\infty) \neq \text{NaN}_i$
$\infty \times 0 \to \Phi$ $(-\infty) \times 0 \to \Phi$ $0 \times \infty \to \Phi$ $0 \times (-\infty) \to \Phi$	$\infty \times 0 = \Phi$ $(-\infty) \times 0 = \Phi$ $0 \times \infty = \Phi$ $0 \times (-\infty) = \Phi$	$\infty \times 0 \to \text{NaN}_i$ $\infty \times (-0) \to \text{NaN}_i$ $(-\infty) \times 0 \to \text{NaN}_i$ $(-\infty) \times (-0) \to \text{NaN}_i$ $0 \times \infty \to \text{NaN}_i$ $(-0) \times \infty \to \text{NaN}_i$ $0 \times (-\infty) \to \text{NaN}_i$ $(-0) \times (-\infty) \to \text{NaN}_i$	$\infty \times 0 \neq \text{NaN}_i$ $\infty \times (-0) \neq \text{NaN}_i$ $(-\infty) \times 0 \neq \text{NaN}_i$ $(-\infty) \times (-0) \neq \text{NaN}_i$ $0 \times \infty \neq \text{NaN}_i$ $(-0) \times \infty \neq \text{NaN}_i$ $0 \times (-\infty) \neq \text{NaN}_i$ $(-0) \times (-\infty) \neq \text{NaN}_i$
$\infty - \infty \to \Phi$	$\infty - \infty = \Phi$	$\infty - \infty \to \text{NaN}_i$	$\infty - \infty \neq \text{NaN}_i$

It has been asked how transarithmetic relates to the Riemann sphere. In general, stereographic projection [9] is the projection of a 3D sphere onto a 2D plane that is tangent to the sphere, or parallel to a plane that is tangent to the sphere. Neumann ([10] foreword p. vi, first footnote) credits the invention of stereographic projection of complex variables to Riemann, and develops a two-sheet representation of it ([10], fourth lecture, pp 131-161). Today, a 3D sphere is called Riemann if it is subjected to a stereographic projection that uniquely carries each point of the sphere, except the projection point at the 'north' pole, onto each point of the complex plane, and which carries the north pole onto a unique point that is not complex, but which lies at an infinite distance from the sphere. In modern parlance, this non-complex point is called *complex infinity.* It is important to note that, in many developments, this point is not assumed to have any properties, other than being infinitely distant from the projection point at the north pole of the sphere. For example, complex infinity is not a complex number, does not have a real or imaginary component, and is not assumed to lie at any specific orientation with respect to, say, the Cartesian-complex co-ordinate-frame. We are perfectly free to take complex infinity identical to undefined. In particular, we may take it to be an actual goldfish riding a unicycle, providing this unlikely object is infinitely distant from the projection point at the north pole of the sphere. All geometrical and topological proofs on the Riemann sphere then go through unaltered. The reader might, therefore, conclude that it is pointless to delve into the internal structure of complex infinity. After all, who cares how a goldfish rides a unicycle or what diameter the wheel is? But stereographic projection can be defined trigonometrically. When these equations are evaluated using transreal or transcomplex arithmetic they can be evaluated everywhere, including at the north pole. Later in the paper, we show that the north pole projects to a point which is infinitely distant at an angle of nullity to the Cartesian-complex co-ordinate frame. The transcomplex structure also has additional parts. It has an axle, which is the non-negative part of the transreal number line, and it has a (unit) circle lying in the plane which contains the north pole and is parallel to the Cartesian-complex (projection) plane. The axle encodes all points that are at a non-negative distance at an angle of nullity, and the circle encodes all points at an infinite distance that lie at a finite (real) angle. Thus, the Riemann sphere is identical to the sphere in the transcomplex case and its projection onto the complex plane is identical. But, in the ordinary complex case the projection of the north pole is not a complex number, whereas in the transcomplex case its projection is a transcomplex number. Thus, projection of the north pole is better behaved in the transcomplex case. The transcomplex case also has additional structure, an axle at angle nullity and a circle at infinity, which the Riemann sphere does not have.

It has been asked if transreal arithmetic is identical to the arithmetic in Beeson and Wiedijk's work on limits in real calculus [24]. The answer is straight forward: a calculus of limits is not any kind of arithmetic. But let us be Charitable and compare functions that map a transreal number onto a transreal number with whatever kind of mappings Beeson and Wiedijk consider, and let us compare transreal limits [5] with whatever limits they consider. Beeson and Wiedijk construct limits using filters on open sets. In transreal arithmetic $\{-\infty\}$, $\{\infty\}$ and $\{\Phi\}$ are closed sets [5], not open sets, so the filter treatment is fundamentally incompatible with transreal arithmetic and transreal calculus. Beeson and Wiedijk explicitly reject the use of ultra filters which would have allowed a direct comparison with our approach. But let us be Charitable and stick to our plan for comparing the systems. Their positive infinity, ∞, is a fixed value with the property $\infty + 1 = \infty$. This is the same as transreal infinity. They also have a negative infinity, $-\infty$, just as transreal arithmetic has. Thus, both approaches agree that $-\infty < r < \infty$ for all real numbers r. However, almost all of their operations on infinities produce results which are different from transreal arithmetic. Firstly, a case that produces the same result. They find that $\lim_{x \to \infty} (1/(x+1)) = 0$. This agrees with the transreal value of the function $1/(\infty + 1) = 0$ and with the transreal limit [5]. But, they find a limit, $\lim_{x \to 0} (x \sin(1/x)) = 0$, where transreal arithmetic evaluates the function as $0 \times \sin(1/0) = 0 \times \Phi = \Phi$ and, in this case, the transreal limit does not exist. Note that it *is* possible to have a transreal limit of nullity, thus $\lim_{x \to a} f(x) = \Phi$ when $f(x) = \Phi$ is a constant function in a neighbourhood around a. See [5] for details. In summary, Beeson and Wiedijk produces results that are fundamentally different from ours. Their approach is more complicated, both in the mathematical techniques used and in the variety of results obtained. We maintain that transreal arithmetic is the simplest possible arithmetic which makes real arithmetic total, so that every arithmetical operation applies to any numbers with the result being a number, and which retains all of the positive results of real calculus. The preservation of the positive results of calculus arises from a topological property. The transreal infinities and nullity have epsilon neighbourhoods evaluated in transreal arithmetic. This arithmetic forces these three, non-finite, points to be path-disconnected from the real numbers [5]. If we attempt to construct any negative example of a limit that is contradictory at a non-finite position or value then we simply note the discontinuity and discard the counter example. On the other hand, if we construct a positive example that does hold at a non-finite position or value then we simply assert that this particular function is continuous at these positions or values. For example, in [6], the transreal exponential is taken to be continuous at $\pm\infty$, but discontinuous at Φ. Quite simply, the topology of transreal numbers, which follows directly from their arithmetical properties, gives us the freedom to accept or reject any non-finite result of real calculus.

It has been asked if transreal arithmetic is identical to any of Carlström's infinite class of arithmetics, each or which allows division by zero [25], and, specifically, if Carlström's number $\perp = 0/0$ is identical to transreal $\Phi = 0/0$. This is a good question. Carlström makes various arithmetics, of semi-rings, total by introducing a reciprocal which is a generalisation of the ordinary reciprocal of rational numbers. Carlström's reciprocal [25] is the same as ours [1], but he uses a different equivalence relation, and different addition and distributivity, which give different arithmetics, all of which are unordered. One can tell if two of Carlström's numbers are equal, but otherwise one cannot have a binary operator which can tell which is the larger number and which is the smaller. Ordering is an important property in practical calculation, including mathematical physics, so Carlström's approach is not readily usable in these applications. As a simple example, imagine the task of weighing out one pound of flour to bake a walnut cake. If the baker shakes exactly one pound of flour into one pan of a balance, with a one pound weight in the other pan, then the baker can see that the scale is balanced; but if the baker puts too much, or too little, flour into the weighing pan then the baker can see that the scale is not balanced, but cannot tell which pan is higher or lower than the other and by what amount. Therefore, the baker cannot estimate how close to the correct weight the flour is, and cannot tell whether to add more flour, or take some out, in order to balance the scale. The physics of such a universe would be hostile in the extreme. No living thing would be able to have any kind of homeostasis, maintaining its temperature, air intake, or whatever. By contrast, transreal arithmetic does have the usual range of relational operators which are based on the fundamental relationships of equality and greater-than [1] [5]. Thus, transreal arithmetic is very different from any of Carlström's arithmetics. As to the second part of the question, note that the zeros is Carlström's number

$\perp = 0/0$ are zeros of a semi-ring, whereas the zeros in transreal $\Phi = 0/0$ are zeros of an ordered field (specifically, zeros of the real numbers). Therefore, transreal $0/0$ has more algebraic structure than Carlström's $0/0$ and is a different number.

It has been asked how to find the non-integral powers of minus infinity. The answer to this question occasioned the development of all of the unpublished transcomplex arithmetics and of the transcomplex arithmetic given in this paper.

We hope we have now demonstrated, to both the general reader and the specialist, that transreal arithmetic is different from other methods for handling division by zero. We maintain that transreal arithmetic is the most useful method for dividing by zero because it uses only pre-existing algorithms of real arithmetic and, thereby, extends the whole of mathematics, computation, and physics in a natural way. We illustrate this, next, with a tutorial, the first part of which has been used, successfully, with secondary school children. This tutorial concludes with a calculation of gravitational and electrostatic singularities by evaluating Newtonian physics with transreal arithmetic. After the tutorial, we show that IEEE floating-point arithmetic can be made more efficient by substituting nullity for minus zero, by re-positioning the codes for positive and negative infinity, and by replacing all of the NaNs with real numbers. We argue that the resulting transfloating-point arithmetic is safer because it has simpler semantics which make it easier to implement programs and to test them. We examine the use of proportions in Newton's *Philosophiae Naturalis Principia Mathematica* and in Euclid's *Elements.* We then introduce the transcomplex numbers and the operations of transcomplex addition, subtraction, multiplication and division. We prove that transcomplex arithmetic contains transreal arithmetic and several ordinary varieties of complex arithmetic. We discuss transcomplex power series, in general, and the transcomplex exponential, in particular. We discuss the transcomplex logarithm. We use the transcomplex exponential and logarithm to define the transcomplex operation of rasing a transcomplex number to a transcomplex power. We discuss the transcomplex Riemann sphere. Finally, we discuss practical applications of transcomplex arithmetic, consider possible future developments, and conclude with a brief summary of what has been achieved in this paper.

3. Tutorial

Transreal arithmetic was invented in 1997 [11] by applying a careful selection of the operations of projective geometry to the point at nullity. Over the subsequent decade, transreal arithmetic was generalised by adding the infinities and making a careful selection of the ordinary algorithms of arithmetic. This selection was axiomatised in 2007 when a machine proof of consistency was given[1]. No further proof of the correctness of the methods is given here. The author has now taught transreal arithmetic to hundreds of people of many ages. The first part of the tutorial works well with secondary school children when the instructor takes care to adapt the order of presentation to suit a student's prior learning and introduces vocabulary only when it is needed. The mathematical vocabulary used here echoes that used in primary schools in England and Wales. It provides a foundation for the introduction of more formal vocabulary. The first part of the tutorial gives the reader the mathematical tools necessary to follow the later parts of the tutorial and the rest of the paper. The second part of the tutorial presents one of the original results of the paper. It shows how to evaluate Newtonian physics using transreal arithmetic to compute the finite resultant force at a singularity where infinitely attractive gravitational forces are opposed by infinitely repulsive electrostatic forces. This calculation is of academic interest in that it allows physical calculations to proceed in the face of division by zero. We break off the analysis at this point and remind the reader that transreal arithmetic gives school children the ability to solve problems involving physical singularities – problems which currently strain or defeat professional physicists who have not learned transreal arithmetic.

3.1 Transreal Arithmetic

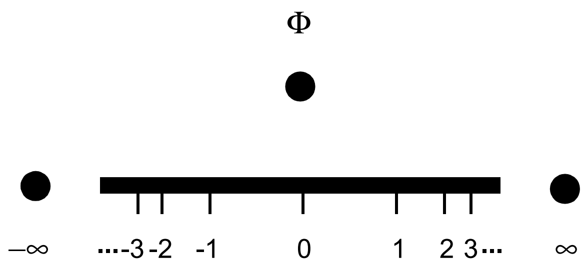


Figure 1: Transreal number-line.

The line, drawn in the figure above, is the *real number-line,* which contains all of the *finite numbers.* These are, firstly, *zero;* secondly, the *counting numbers* (positive integers); thirdly the *negative counting numbers* (negative integers); and the *measuring numbers.* The measuring numbers are, firstly, *fractions,* being rational numbers, excluding the integers and specifically excluding zero; secondly, irrational numbers. Pedagogues will note that the arithmetic of fractions taught in primary and secondary schools is a proper subset of rational arithmetic. Most significantly, zero is not a school-fraction. For example, school students write $1/2 - 1/2 = 0$, where 0 is an integer. They are not taught to apply the rules of rational arithmetic which give $1/2 - 1/2 = (1-1)/2 = 0/2 = 0/1 = 0$. The big dots, ●, show the *non-finite numbers.* The *extended-real number-line* is the real number line with infinity, ∞, and minus infinity, $-\infty$. Every number to the right of zero, on the extended-real number-line, is *positive;* every number to the left of zero, on the extended-real number-line, is *negative.* Zero is neither positive nor negative. The sign of zero is zero. This definition, which is widely accepted in the world, contradicts the teaching of sign in French speaking primary and secondary schools, where it is taught that zero is a positive number. The *transreal number-line* is the extended-real number-line with nullity. Nullity, Φ, lies off the extended-real number-line and lies at the angle nullity to it. Nullity is neither positive nor negative. The sign of nullity is nullity. Infinity, ∞, is the most positive number and minus infinity, $-\infty$, is the most negative number. Infinity is bigger (further to the right on the extended-real number-line) than any number, except itself and nullity. Minus infinity is smaller (further to the left on the extended-real number-line) than any number except itself and nullity. Nullity is equal to itself, but is not bigger than or smaller than any number – because it is not on the number line. This paints a mental picture of how the transreal numbers relate to each other and gives a vocabulary for talking about these relationships. Both aspects are extended in more advanced study.

The *canonical* or *standard* or *least terms* form of certain numbers is as follows. Transreal one, 1, is real one divided by (over) real one: $1 = 1/1$. Transreal minus-one, -1, is real minus-one divided by (over) real one: $-1/1$. Transreal zero, 0, is real zero divided by real one: $0 = 0/1$. *Transreal infinity,* ∞, is real one divided by (over) real zero: $\infty = 1/0$.

Transreal minus-infinity, $-\infty$, is real minus-one divided by (over) real zero: $-\infty = (-1)/0$. Transreal *nullity,* Φ, is real zero divided by (over) real zero: $\Phi = 0/0$. Any irrational number x is x divided by (over) real one: $x = x/1$.

Transreal numbers can be expressed as *transreal fractions,* n/d, of a real *numerator,* n, and a non-negative, real *denominator,* d. Transreal numbers with a non-finite numerator or denominator simplify to this form. An *improper fraction* can be written with a negative denominator, but it must be converted to a *proper fraction,* by carrying the sign up to the numerator, n, before applying any of the transreal arithmetical operations. This can be done by multiplying both the numerator and denominator by minus one; it can be done by negating both the numerator and the denominator, using subtraction; and it can be done, instrumentally, by moving the minus sign from the denominator to the numerator.

$$\frac{n}{-d} = \frac{-1 \times n}{-1 \times (-d)} = \frac{-n}{-(-d)} = \frac{-n}{d} \tag{1}$$

Transreal infinity is equal to any positive number divided by zero. Transreal minus-infinity is equal to any negative number divided by zero. Zero is equal to zero divided by any positive or negative number. With $k > 0$ we have:

$$\infty = \frac{1}{0} = \frac{k}{0} \qquad -\infty = \frac{-1}{0} = \frac{-k}{0} \qquad 0 = \frac{0}{1} = \frac{0}{k} = \frac{0}{-k} \tag{2}$$

The ordinary rules for multiplication and division apply universally to the proper transreal-fractions. That is, these rules apply without side conditions. In particular, division by zero is allowed.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \tag{3}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} \tag{4}$$

We can now calculate one of the formulas given in the introduction:

$$\frac{\infty}{\infty} = \infty \div \infty = \frac{1}{0} \div \frac{1}{0} = \frac{1}{0} \times \frac{0}{1} = \frac{1 \times 0}{0 \times 1} = \frac{0}{0} = \Phi \tag{5}$$

A different example illustrates the procedure for keeping the sign of a transreal fraction in the numerator. Here $\infty \div (-3) = -\infty$. Without this procedure the result would be ∞, which would violate the usual rule that the product of a positive and a negative number is negative.

$$\infty \div (-3) = \frac{1}{0} \div \frac{-3}{1} = \frac{1}{0} \times \frac{1}{-3} = \frac{1}{0} \times \frac{-1}{3} = \frac{1 \times (-1)}{0 \times 3} = \frac{-1}{0} = -\infty \tag{6}$$

Addition is more difficult than multiplication and division because it breaks into two cases: the addition of two signed infinities and the general case. Two infinities are added using the ordinary rule for adding two fractions with a common denominator:

$$(\pm\infty) + (\pm\infty) = \frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0} \tag{7}$$

Finite fractions may be added using this rule, if they happen to have a common denominator, but infinities cannot be added using the following general rule of addition. If infinities were added by the general rule we would have $\infty + \infty = \Phi$, but this is inconsistent with various arithmetics of the infinite that have $\infty + \infty = \infty$. See, for example: [26] [27] [28] [29].

The general case of addition is:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d + b \times c}{b \times d} \tag{8}$$

Subtraction is the addition of a negated number:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} \tag{9}$$

We can now calculate one of the formulas given in the introduction:

$$\infty - \infty = \frac{1}{0} - \frac{1}{0} = \frac{1}{0} + \frac{-1}{0} = \frac{1 + (-1)}{0} = \frac{1 - 1}{0} = \frac{0}{0} = \Phi \tag{10}$$

A different example illustrates general addition. If the rule for adding two infinities were used here the result would be $-\infty$. This would lead to a more complicated arithmetic that is a bigger departure from ordinary arithmetic and would make it more difficult to use nullity to model an unknown value. As things stand in transreal arithmetic: finite numbers can be used to model quantities whose magnitude and sign are know completely; infinities can be used to model quantities whose sign is known completely, but whose magnitude is known only to be big; and nullity can be used to model quantities where nothing is known about their sign or magnitude. Ultimately, the transreal definition of arithmetic on nullity is an aesthetic choice which might be overturned by experience.

$$\Phi - \infty = \frac{0}{0} - \frac{1}{0} = \frac{0}{0} + \frac{-1}{0} = \frac{0 \times 0 + 0 \times (-1)}{0 \times 0} = \frac{0}{0} = \Phi \tag{11}$$

Transreal arithmetic is totally associative, totally commutative, but is only partially distributive at infinity. When we present proofs later in the paper we give some computations explicitly and assume that the reader can supply all similar associative and commutative cases. This is an ordinary assumption of proofs given in many parts of mathematics; though some formal work, and all machine proofs, give all cases explicitly.

The axiom of transreal distributivity [1] can be broken down into a number of cases. As usual, a number, a, distributes over $b + c$ when:

$$a(b + c) = a \times b + a \times c \tag{12}$$

If a is a finite number or nullity then a distributes over any $b + c$. If a is infinity or minus infinity then a distributes if $b + c = \Phi$ or $b + c = 0$ or b and c have the same sign. Two numbers have the same sign if they are both positive, both negative, both zero, or both nullity. For example, $\infty(2-2)$ is distributive because:

$$\begin{aligned}\infty(2-2) &= \frac{1}{0}\times\left(\frac{2}{1}-\frac{2}{1}\right) = \frac{1}{0}\times\left(\frac{2}{1}+\frac{-2}{1}\right) = \frac{1}{0}\times\left(\frac{2+(-2)}{1}\right) = \frac{1}{0}\times\left(\frac{2-2}{1}\right) = \frac{1}{0}\times\frac{0}{1} = \frac{1\times 0}{0\times 1} \\ &= \frac{0}{0} = \Phi\end{aligned} \tag{13}$$

and

$$\begin{aligned}\infty\times 2-\infty\times 2 &= \frac{1}{0}\times\frac{2}{0}-\frac{1}{0}\times\frac{2}{0} = \frac{1\times 2}{0\times 0}-\frac{1\times 2}{0\times 0} = \frac{2}{0}-\frac{2}{0} = \frac{1}{0}-\frac{1}{0} = \frac{1}{0}+\frac{-1}{0} = \frac{1+(-1)}{0} \\ &= \frac{1-1}{0} = \frac{0}{0} = \Phi\end{aligned} \tag{14}$$

Conversely, $\infty(2-1)$ is non-distributive, because:

$$\begin{aligned}\infty(2-1) &= \frac{1}{0}\times\left(\frac{2}{1}-\frac{1}{1}\right) = \frac{1}{0}\times\left(\frac{2}{1}+\frac{-1}{1}\right) = \frac{1}{0}\times\left(\frac{2+(-1)}{1}\right) = \frac{1}{0}\times\left(\frac{2-1}{1}\right) = \frac{1}{0}\times\frac{1}{1} = \frac{1\times 1}{0\times 1} \\ &= \frac{1}{0} = \infty\end{aligned} \tag{15}$$

but

$$\begin{aligned}\infty\times 2-\infty\times 1 &= \frac{1}{0}\times\frac{2}{0}-\frac{1}{0}\times\frac{1}{1} = \frac{1\times 2}{0\times 0}-\frac{1\times 1}{0\times 1} = \frac{2}{0}-\frac{1}{0} = \frac{1}{0}-\frac{1}{0} = \frac{1}{0}+\frac{-1}{0} = \frac{1+(-1)}{0} \\ &= \frac{1-1}{0} = \frac{0}{0} = \Phi\end{aligned} \tag{16}$$

These examples are formal because they apply the formal definitions given above; but, in practice, one soon comes to take effective shortcuts. For example:

$$\infty(2-2) = \infty\times 0 = \Phi \text{ and } \infty\times 2-\infty\times 1 = \infty-\infty = \Phi \tag{17}$$

These particular shortcuts are just applications of the axioms of transreal arithmetic [1]. In fact, the axiomatisation was obtained as an axiomatisation of shortcuts. This is both an advantage, in that the axioms are concise, and a disadvantage, in that the axioms *conceal* algorithmic structure. It should be remembered that the axioms were devised in order to obtain a machine proof of the consistency of transreal arithmetic; but it would be possible to develop a more leisurely axiomatisation of the algorithms, along the lines of the successive axiomatisation of natural numbers, leading to integers, leading to rational numbers, leading to real numbers, leading to complex numbers, as given, for example, in [30].

This is as much of the tutorial as has been presented to school children. We have found that children in the age range from 12 to 16 years, inclusive, have no psychological barriers to learning transreal arithmetic, but that older school children, who have started the A-level syllabus, do have such intellectual inhibitions. Adults are particularly resistant to the notion of dividing by zero.

There is a great deal more to say about transreal arithmetic, which secondary-school students could learn; but we already have enough to extend secondary-school physics lessons to the calculation of Newtonian forces at, and near, a singularity.

3.2 Newtonian Singularity

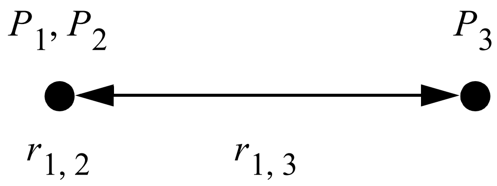


Figure 2: Particles at, and near, a singularity.

Consider three particles, P_i, at finite locations, as shown in the figure above. P_1 and P_2 lie at a distance $r_{1,2} = 0$ from each other and, therefore, form a singularity. P_3 lies at a distance, $r_{1,3}$, of one Planck length from the singularity. Suppose that the particles are heavy electrons (tau electrons, obeying the Pauli exclusion principle at the singularity). Use the following approximations: $r_{1,3} = 1.6 \times 10^{-35}$ metres; the mass of each tau electron is $m_i = 3.2 \times 10^{-27}$ kilo grammes; each tau electron carries a negative charge of $q_i = 1.6 \times 10^{-19}$ Coulombs; the gravitational attraction between two masses is $F = ((6.7 \times 10^{-11})m_1m_2)/r^2$ Newtons; and the electrostatic force between two charges is $F = ((9 \times 10^9)q_1q_2)/r^2$ Newtons.

Calculate the gravitational attraction between P_1 and P_2 at the singularity. Thus:

$$F = \frac{(6.7 \times 10^{-11})m_1m_2}{r^2} = \frac{(6.7 \times 10^{-11}) \times (3.2 \times 10^{-27}) \times (3.2 \times 10^{-27})}{0^2} = \frac{1}{0} = \infty \tag{18}$$

Calculate the gravitational attraction, in Newtons, between the singularity and P_3.

$$\begin{aligned} F &= \frac{(6.7 \times 10^{-11})m_1m_2}{r^2} = \frac{(6.7 \times 10^{-11}) \times (2 \times 3.2 \times 10^{-27}) \times (3.2 \times 10^{-27})}{(1.6 \times 10^{-35})^2} \\ &= \frac{6.7 \times 2 \times 3.2 \times 3.2 \times 10^{-11} \times 10^{-27} \times 10^{-27}}{2.56 \times 10^{-70}} \cong \frac{137.21 \times 10^{-65}}{2.56 \times 10^{-70}} \cong 53.6 \times 10^5 \cong 5.4 \times 10^6 \end{aligned} \tag{19}$$

Similarly, calculate the electrostatic repulsion of ∞ Newtons between P_1 and P_2 at the singularity, and of, approximately, 1.8×10^{42} Newtons between the singularity and P_3.

Now calculate the resultant force of $\infty - \infty = \Phi$ Newtons at the singularity, and of $5.4 \times 10^6 - 1.8 \times 10^{42} \cong -1.8 \times 10^{42}$ Newtons at P_3. Thus, the resultant force is large and repulsive at P_3; but, in order to interpret the force at the singularity, we need to know how the laws of motion operate in the presence of non-finite quantities.

We adopt the debating stance that the laws of motion, as stated by Sir Isaac Newton, can be read so as to apply to non-finite quantities. The first law, stated in Latin, may be translated as, "Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed." See [12] p. 416. Whereas nullity lies outside the extended-real universe, and lies at no real orientation to it, as shown in Figure 1, we accept, as an axiom, that it cannot impress any force on a body in the extended-real universe. In other words, a force of nullity has the same influence, in the extended-real universe, as a force of zero. At the point at nullity itself, a force of nullity operates according to transarithmetic. This differs from modern formalisms of the first law, in that they allow an alteration of state only in response to a non-zero, resultant force, where we allow an alteration of state only in response to a force which is both non-zero and non-nullity. In other words, in response to an infinite or else non-zero, finite force. We differ, too, in allowing infinite forces.

The difference between infinite and nullity forces may best be understood by considering the following diagram and a modification of the argument given in [5].

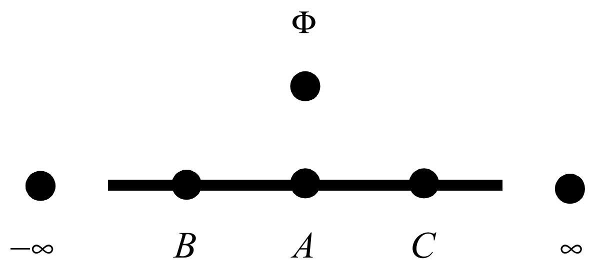


Figure 3: Comparison of infinite and nullity forces.

Consider any point, A, on the real number line. There is a point, B, on the real number line, that is intermediate between A and $-\infty$. There is also a point, C, on the real number line, that is intermediate between A and ∞. A force of any non-zero, real magnitude can move a body at A to B, that is toward $-\infty$, or else to C, that is toward ∞, depending on the sign of the force. Thus, a body may experience a continuous motion resulting from a Newtonian force that moves it, without finite bound, toward a signed infinity. We suppose that this Newtonian force can be extended to infinite magnitude so that it moves A exactly to a point at infinity, marked $-\infty$ or ∞ in the diagram. By contrast, there is no point intermediate between A and Φ so no force whatever can move a body continuously from A in the direction of Φ. In particular, no Newtonian force can move a body in this way. We suppose that there is no extension of this, non-existent, Newtonian force that operates on bodies in the real-numbered universe. Similarly, there is no point intermediate between $-\infty$ and Φ, nor between ∞ and Φ, so no force whatever can move a body continuously from $-\infty$ or ∞ in the direction of Φ. In particular, no Newtonian force can move a body in this way. We suppose that there is no extension of this, non-existent, Newtonian force that operates on bodies in the extended-real universe. So far as a body in the extended-real universe is concerned, a force of nullity is indistinguishable from a force of zero so we shall say that nullity forces project into the extended-real universe as zero forces. By what has just been said about betweenness, a body at Φ cannot be moved continuously toward any point in the extended-real universe, but as all transreal or transcomplex quantities combine arithmetically with Φ to produce Φ, we may say that a nullity force operates according to the rules of transarithmetic and leaves a body at Φ unmoved.

Notice that the above argument is not a proof. It seeks only to make an axiom seem plausible. The axiom that nullity forces project into the extended-real universe as zero forces, along with the entire argument, would be thrown out if there were any good reason, especially an empirical reason, to believe that some other state of affairs applies in the universe we live in.

The second law, stated in Latin, may be translated as, "A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed." See [12] p. 416. Reading "a change in motion" as "a change in the quantity of motion," and reading this, as a change in momentum, makes force the subject of the law, as stated in the form: $F = m \times a$. (It should be noted, however, that reading modern physical concepts into Newton's work can be problematical. A modern reading involving differentials might be preferred. For our purposes, however, the form just given is adequate.) But if all of the parameters of acceleration, a, force, F, and momentum, m, are transreal then there are some combinations of two parameters that prevent the third parameter from being computed. For example, if an infinite force acts on a body with zero mass then we cannot compute the acceleration in $\infty = 0 \times a$. This difficulty is overcome, here, by making each parameter the subject of an equation. Thus, we also have $a = F/m$ and $m = F/a$. Using the former, we have $a = \infty/0 = \infty$ so that an infinite force impresses an infinite acceleration on a body with zero mass. In this case the momentum is $m = \infty/\infty = \Phi$ and, by analogy with our rule for forces, we say that a nullity momentum operates as a zero momentum on a body anywhere in the extended-real universe. More generally, we say that if an equation deals with a body in the extended-real universe, and the equation involves any force term, and the equation evaluates to nullity, then the result of nullity may be replaced by zero. The rule for re-writing nullity as zero in the extended-real universe is a physical rule, not a general, mathematical, one. With this understanding, it follows that all bodies with zero Newtonian mass have zero Newtonian momentum. (Proof: with the usual notation, $p = mv = 0 \times v$, but $0 \times v = 0$ for all finite, v, and $0 \times v = \Phi \to 0$ for all non-finite v.)

We restate our second law as, "A motion obeys all satisfied equations of the form $F = m \times a$, $a = F/m$, or $m = F/a$ where a is a transcomplex (vector) acceleration, F is a transcomplex (vector) force, m is a transreal (scalar) mass, and the directions of a and F are equal." The terms *acceleration, force,* and *mass* have their usual, modern definitions; but are permitted to take on transvalues.

The third law, stated in Latin, may be translated as, "To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction." See [12] p. 417. We restate this law by making the arithmetic of actions and reactions explicit as transcomplex (vector) operations on forces. Thus, "To any action, F, there is always an opposite and equal reaction, $-F$; in other words, the actions of two bodies upon each other are always equal and always opposite in direction."

In our restatement of the laws of motion, all of a, F, m are transcomplex values, but can be read as transreal values, laid off in real directions as specified by Newton, or else laid of in the direction nullity. Thus, both transcomplex arithmetic and transreal vector-arithmetic, support a Newtonian transphysics which operates at singularities.

We tabulate the three equations, used to redefine the second law, for all of their transreal parameters. The values k_i in the table are arbitrary, real, positive constants. Recall that any value of Φ in the body of the table can be re-written as zero only if the physical body is in the extended-real universe. Values of nullity in the heading row and column cannot be rewritten. For example, a body that lies at nullity may have a mass of nullity.

Now it is clear that the tau electron P_3, near the singularity, is repelled from the singularity with considerable force, of magnitude, approximately, 1.8×10^{42} Newtons; but P_1 and P_2, at the singularity, have a force of nullity acting on them, which is equivalent to a force of zero, so they remain at the singularity, being neither mutually repelled nor attracted.

Table 3: $F = m \times a$ with $0 < k_i < \infty$

$F = m \times a$		a					
		$-\infty$	$-k_a$	0	k_a	∞	Φ
m	0	Φ	0	0	0	Φ	Φ
	k_m	$-\infty$	$-k_m k_a$	0	$k_m k_a$	∞	Φ
	∞	$-\infty$	$-\infty$	Φ	∞	∞	Φ
	Φ	Φ	Φ	Φ	Φ	Φ	Φ

Table 4: $a = F/m$ with $0 < k_i < \infty$

$a = F/m$		m			
		0	k_m	∞	Φ
F	$-\infty$	$-\infty$	$-\infty$	Φ	Φ
	$-k_F$	$-\infty$	$-k_F/k_m$	0	Φ
	0	Φ	0	0	Φ
	k_F	∞	k_F/k_m	0	Φ
	∞	∞	∞	Φ	Φ
	Φ	Φ	Φ	Φ	Φ

Table 5: $m = F/a$ with $0 < k_i < \infty$

$m = F/a$		a					
		$-\infty$	$-k_a$	0	k_a	∞	Φ
F	$-\infty$	Φ	∞	$-\infty$	$-\infty$	Φ	Φ
	$-k_F$	0	k_F/k_a	$-\infty$	$-k_F/k_a$	0	Φ
	0	0	0	Φ	0	0	Φ
	k_F	0	$-k_F/k_a$	∞	k_F/k_a	0	Φ
	∞	Φ	$-\infty$	∞	∞	Φ	Φ
	Φ	Φ	Φ	Φ	Φ	Φ	Φ

3.3 Discussion

The reader is warned that there is a great deal more to know about transreal arithmetic and its applications than is presented in this brief tutorial. In particular, we have yet to describe the proper treatment of distance and orientation, which we do in the development of transcomplex arithmetic. Nonetheless, the reader may now compute the resultant of any forces operating on particles at a singularity, or at any fixed, positive, distance from the singularity, providing all of the locations are finite. Later on, when non-finite distance and orientation are understood, it will be possible to compute these properties at any location.

A simpler exercise, which builds confidence, is to confirm the results in the left hand part of Table 2 and all of the results in Tables 3-5. The reader is reminded that all of the numerical examples in this paper are computed by software in the on-line appendix. The software follows a transcomplex computational path, wherever possible, and agrees with all of the examples in this paper, which the author prepared by hand, and which the reader can check by hand. In particular, this demonstrates that all of the examples of transreal arithmetic are obtained by a corresponding transcomplex computation. Checking all of the transcomplex computations by hand is laborious, but the reader might care to try a few examples once transcomplex arithmetic has been presented.

4. Floating-Point Arithmetic

In this section we show that floating-point arithmetic is more efficient and safer when it is based on transreal arithmetic, rather than when it uses the IEEE specification [3] [4] which employs objects that are not a number, NaN, and minus zero, -0. This is offered both as an improvement to computer arithmetic and as a demonstration that transreal arithmetic is useful. It also places floating-point implementations of both complex and transcomplex arithmetic on a firmer footing. We note that there is, as yet, no international standard for the execution of complex arithmetic on a digital computer.

4.1 IEEE Floating-Point Format

A binary, IEEE floating-point number is represented as follows. See [3], especially p. 7-9. A floating-point number is represented by three bit-strings s, e, f being: a 1-bit sign, s; a biased exponent, $e = (E+B) \geq 0$, where E is the unbiased (signed) exponent and B is the bias; and a fraction, $f = b_0 \bullet b_1 b_2 \ldots b_{p-1}$, where the b_i are bits and p is the number of significand bits (vernier precision). The range of the unbiased exponent, E, includes every integer between E_{min} and E_{max}, inclusive, and also has two other reserved values: $E_{\text{max}} + 1$ is used to encode $\pm\infty$ and the *N*ot *a* *N*umber, NaN, objects; and $E_{\text{min}} - 1$ may be used to encode ± 0 and the denormalised numbers (explained next), though it may remain an option to use some other exponent to encode these zeros and denormal numbers. For example, $E_{\text{max}} + 2$ may be so used. (The standard [3] is ambiguous on this point: p. 8 allows the option, p. 9 disallows it. The computing industry has adopted $E_{\text{min}} - 1$.) But, whatever exponent is reserved, all of the representable real numbers are of the form $(-1)^s 2^E (b_0 \bullet b_1 b_2 \ldots b_{p-1})$ which is redundant as $2^0(1 \bullet 0) = 2^1(0 \bullet 1) = 2^2(0 \bullet 01) = \ldots$ However, the standard specifies that non-extended precisions use non-overlapping normalised and denormalised numbers. Normalised numbers have $d_0 = 1$ with the exponent variable. Denormalised numbers have $d_0 = 0$ with the exponent a reserved value. This scheme of normal and denormal numbers has no redundancy. However, the extended precision formats are allowed to take d_0 and the exponent arbitrarily, within the scheme, so they may contain the redundancy. There are three objects which are not real numbers and not NaNs encoded by the bit strings. These are: -0, $-\infty$, ∞. In addition, there are at least two NaNs, a quiet NaN and a signalling NaN, and at most $2^p - 1$ NaNs. Note that the sign bit, s, is not defined to carry any information when the bit strings e and f together encode a NaN object. In this case we say that the NaNs are *unsigned.* When we pay attention to the sign bit, in order to count the number of wasted states, we say that the NaNs are *signed.* There are $2(2^p - 1) = 2^{p+1} - 2$ signed NaN states, only half of which are distinguished by the standard, and only two of which must be implemented. The number of wasted states is tabulated below, using the nomenclature of the 2008 version of the standard [4].

Table 6: Wasted NaN states in IEEE floating-point arithmetic

Name	**Common Name**	**p**	**Wasted NaN States**
binary16	half precision	10	2 046
binary32	single precision	23	16 777 214
binary64	double precision	52	9 007 199 254 740 990
binary128	quadruple precision	112	10 384 593 717 069 655 257 060 992 658 440 190

The wasted NaN states are free to be re-assigned when the floating-point format models transreal arithmetic – rather than modelling real arithmetic, as extended by the IEEE standards. And, as transreal arithmetic has $0 \equiv -0$, the floating-point state for IEEE -0 can also be reassigned in transfloating-point arithmetic. In summary, the IEEE floating-point format has a great many redundant states which can be re-assigned to unique real numbers in transfloating-point arithmetic to produce an irredundant representation.

4.2 Transfloating-point Format

Transreal numbers [1] are the real numbers augmented with three non-finite numbers: $-\infty$, ∞, Φ. Transreal arithmetic has no object -0, that is distinct from zero, so we re-assign the IEEE bit-pattern for -0 to Φ. Henceforth, nullity, zero, and the denormal numbers are all represented by one of the two reserved exponents, usually $E_{\min} - 1$. We do not reserve the exponent $E_{\max} + 1$. Instead, we re-assign it to the ordinary exponent bits, thereby exactly doubling the range of real numbers encoded by the scheme. However, we do reserve the most positive bit pattern, of the form $2^{(E_{\max} + 1)}(1_0 \bullet 1_1 1_2 \ldots 1_{p-1})$, to represent ∞. Similarly we reserve the most negative bit pattern, of the form $-(2^{(E_{\max} + 1)})(1_0 \bullet 1_1 1_2 \ldots 1_{p-1})$, to represent $-\infty$. Thus, we reserve two bit patterns with exponent $E_{\max} + 1$ and say that we have nearly doubled the numerical range of the real numbers represented by the scheme. Alternatively, if we increment the bias, B, by one, then we retain the same range of real numbers, less one signed number, differing from the most extreme, real, IEEE floating-point number in only its least significant bit, but we thereby extend the scale precision by exactly one binade.

In summary, we keep the IEEE encodings of 0 and of the denormalised numbers and of the normalised numbers, but we re-map the code for IEEE -0 to Φ, and we re-position $\pm\infty$ so that the significand, with maximal exponent, has all bits set rather than clear. We then re-allocate all of the remaining bit patterns in this significand to normalised, and therefore unique, real numbers. Thus all of the NaNs are replaced by unique numbers. This removes all redundancy from the IEEE non-extended formats. Therefore, the transfloating-point format is more efficient than the IEEE format. We leave the reader to decide whether it is better to keep the IEEE bias, which almost doubles the arithmetical range of real numbers, or else to increment the IEEE bias by one, which leaves the range almost the same, but extends the scale precision by exactly one binade.

4.3 Relational Operators

The IEEE standard ([3] pp. 12-13) provides four, mutually exclusive, Boolean, ordering relations: *less than* (<), *equal* (=), *greater than* (>), and *unordered* (?). As special cases, *minus zero* and *zero* compare equal ($-0 = 0$), even though these two objects are different, and *NaN* objects compare unequal, even if they are identical $\mathrm{NaN}_i \neq \mathrm{NaN}_i$. Apart from these special cases, the relations *less than, equal,* and *greater than* all have their usual mathematical meanings. The *unordered* relation is true (*T*) if any of its arguments is NaN, and is false (*F*) otherwise. This gives the only standard way of determining if an object, x, is NaN: by testing the truth of $x?x$. The forms *isnan*(x) and $x \neq x$ are specifically excluded from the standard ([3] p 17). While the four ordering relations are mutually exclusive, they are not orthogonal: there are 14 positive relations that have no NOT predicate, and 12 negations, which do contain a NOT predicate. The non-negated pair of relations is *equal* (=) and *not equal* (?<>). Consequently, the missing negations are NOT(=) and NOT(?<>). Precisely 12 of the Boolean relations generate exceptions (error conditions) if any of their arguments is NaN. But the implementor of the standard is free to choose whether to supply Boolean operations, with exceptions, or else flags – *greater, less, equal, unordered* – without exceptions. If Boolean relations are implemented then only the first 6 are mandatory (=, ?<>, >, >=, <, <=). If exceptions are generated, the programmer can choose whether to handle the exceptions in a trap, or else to let the standard complying system follow its default behaviour. Thus, the IEEE standard specifies a complicated ordering of floating-point numbers and allows considerable variation in how exceptions are implemented.

Table 7: IEEE ordering relations: 14 positive, 12 negations, 12 exceptions

Predicate	**Greater**	**Less**	**Equal**	**Unordered**	**Exception**
=	F	F	T	F	No
?<>	T	T	F	T	No
>	T	F	F	F	Yes
>=	T	F	T	F	Yes
<	F	T	F	F	Yes
<=	F	T	T	F	Yes
?	F	F	F	T	No
<>	T	T	F	F	Yes
<=>	T	T	T	F	Yes
?>	T	F	F	T	No
?>=	T	F	T	T	No
?<	F	T	F	T	No
?<=	F	T	T	T	No
?=	F	F	T	T	No
NOT(>)	F	T	T	T	Yes
NOT(>=)	F	T	F	T	Yes
NOT(<)	T	F	T	T	Yes
NOT(<=)	T	F	F	T	Yes
NOT(?)	T	T	T	F	No
NOT(<>)	F	F	T	T	Yes
NOT(<=>)	F	F	F	T	Yes
NOT(?>)	F	T	T	F	No
NOT(?>=)	F	T	F	F	No
NOT(?<)	T	F	T	F	No
NOT(?<=)	T	F	F	F	No
NOT(?=)	T	T	F	F	No

Transreal arithmetic [1] provides three, mutually exclusive, Boolean, ordering relations: *less than* (<), *equal* (=), and *greater than* (>). These relations have their usual mathematical meaning. There are no special cases and no exceptions. The operations are orthogonal, with no missing predicates and no missing negations. The empty symbol, with no occurrences of <, =, > is not listed because it is empty. Nonetheless, this symbol could be supported by a computer language, if desired. The implementor is free to implement transreal ordering relations with Boolean predicates, flags, or any sufficient method. No exception handling is needed. Our preference is to indicate negations by a character, such as (!), rather than writing out NOT as a predicate. Thus, we prefer to use, respectively, !=, !>, !>=, !<, !<=, !<>, !<=> in preference to NOT(=), NOT(>), NOT(>=), NOT(<), NOT(<=), NOT(<>), NOT(<=>). All of the relational operators are distinct, as can be seen from the fact that no two rows, in the body of the table, are equal. Every operator can return either true or else false, depending on its arguments. For example $\Phi <=> 0 \rightarrow \text{false}$ and $\Phi\ !<=> 0 \rightarrow \text{true}$. This relation may be used to determine if exactly one of its arguments is nullity.

Table 8: Transreal ordering relations: 6 positive, 6 negations, 0 exceptions

Predicate	**Greater**	**Less**	**Equal**
=	F	F	T
>	T	F	F
>=	T	F	T
<	F	T	F
<=	F	T	T
<>	T	T	F
<=>	T	T	T
!=	T	T	F
!>	F	T	T
!>=	F	T	F
!<	T	F	T
!<=	T	F	F
!<>	F	F	T
!<=>	F	F	F

In summary, transreal arithmetic provides the usual mathematical ordering relations, with their ordinary mathematical meanings. But IEEE floating-point arithmetic provides an additional (and entirely redundant) mathematical ordering relation (*unordered*), changes the meaning of the ordinary mathematical relation of equality, changes the other relational operations so that some of them generate exceptions, omits two negations, forbids the implementor from using both flags and Boolean relations, and, perversely, recommends that NaNs are identified by non-standard methods. All of these departures from the ordinary mathematical ordering relations are redundant, as the counter example of transreal arithmetic demonstrates.

We suggest that there are very few programmers who know all of the IEEE ordering relations and can apply them correctly in their various implementations. Even where such programmers are employed, say on the implementation and testing of safety critical systems, the IEEE standard [3] gives little guidance on how NaNs arise in mathematical functions. By contrast, nullity is a number so its occurrence in mathematical functions can be deduced or else defined. In conclusion, we suggest that the IEEE standard is so complicated, and is so mathematically perverse, that it encourages programmer error, making the standard dangerous.

4.4 Discussion

IEEE floating-point arithmetic uses a total system of bits to represent real numbers. The system is total in that every bit can be set, independently, to every possible value: zero or else one. In theory, it is possible to construct a bijection between any two sets of equal cardinality but, in practice, it seems to be difficult to map a partial system onto a total one without wasting states or creating error states. This is certainly the case when a finite subset of real numbers, a partial system, is mapped onto a total floating-point format. For example, a sign bit has two states, but real numbers have three signs: negative, zero, and positive. If we are to encode all three signs in two bits then there is one wasted state. IEEE floating-point arithmetic avoids this waste by introducing two classes of numbers: normal and denormal. The sign bit of a normal number indicates whether it is strictly positive or else strictly negative. The sign bit of a denormal number denotes wether it is positive, including zero, or else negative, including negative zero – with negative zero being specially defined in IEEE floating-point arithmetic. By contrast, there are four signs in transreal arithmetic: negative, zero, positive, and nullity. Transfloating-point arithmetic takes normal and denormal numbers strictly positive or else strictly negative, and takes the zero significand as real zero or else transreal nullity. No new class of object need be invented in order to use all of the available sign states. The position is even more stark on division by zero. Real arithmetic cannot divide by zero so states must be used somewhere in a floating-point unit to describe the errors that arise when the programmer instructs a division by zero. IEEE floating-point arithmetic uses at least four states to handle these errors: *negative infinity, positive infinity, quiet NaN,* and *signalling NaN.* None of these are numbers. The NaNs are explicitly NOT A NUMBER ([24], page 8) and the infinities are defined to be limits, where these limits exist ([24], page 13). This definition is not Turing computable so we must read it as an analogy that is supposed to justify the non-finite arithmetic defined in the rest of the standard. By contrast, *positive infinity, negative infinity,* and *nullity* are identified as particular ratios of integers in transreal arithmetic, such ratios obeying ordinary algorithms of rational arithmetic. (Specifically, $\infty = 1/0$; $-\infty = -1/0$; $\Phi = 0/0$.) Thus, transreal arithmetic adds three non-finite numbers to the real numbers modelled by the floating-point format, whereas IEEE arithmetic adds a very large number of NaN states – half of the signed NaN states being explicitly wasted. As we said, it is not logically necessary that representing a partial system in a total one wastes states or creates error states but, in practice, this does seem to be a common outcome in practical computer systems. This observation justifies the study of total systems of computation.

The totality of transfloating-point arithmetic brings further advantages. IEEE floating-point arithmetic defines five classes of exception (error). See [24], pp. 8, 14-15. These are: *inexact, underflow, overflow, divide by zero,* and *invalid operation.* In transreal arithmetic division by zero is not an exception so the only value in signalling it would be for backward compatibility. Similarly, a total arithmetic can support total functions so one can design computer systems with no invalid operations (and we have done this [5]). This leaves just the exceptions: inexact, underflow, and overflow. These three exceptions arise only from numerical round-off. But in IEEE's default rounding mode, *round to nearest, ties to even* ([24], page 10), the position is simpler than this. Underflow occurs if and only if an inexact operation produces the result zero, and overflow occurs if and only if an inexact operation produces a signed infinity as result. Therefore, in this rounding mode, only one exception is needed: inexact. One can then test the result to see if it is zero (underflow), a non-zero-real number (inexact), or an infinity (overflow). Conversely, if there is no inexact exception then the result is exactly: zero, non-zero-real, an infinity, or nullity. Thus, transfloating-point arithmetic removes two of the five exceptions in all rounding modes and removes four of the five exceptions in the default rounding mode. This reduction in complexity makes it simpler to implement correct programs, which is to say that transfloating-point arithmetic is, again, safer than IEEE floating-point arithmetic.

We leave the reader to decide whether it would be useful to reserve one bit of a transfloating-point format to be an inexact flag.

In conclusion, we have proved that transfloating-point arithmetic is more efficient than IEEE floating-point arithmetic because it has no redundant states. We suggest, further, that the IEEE standard is dangerous for both a psychological and a technical reason. Firstly, it is so complicated that it encourages programmer error. Secondly, it has exceptional, that is, error, states which can terminate execution abnormally. By contrast, transreal arithmetic is simple and has no exceptional states so it can never terminate abnormally. Therefore, transfloating-point arithmetic is both more efficient and safer than IEEE floating-point arithmetic. However, transreal arithmetic is new. Very few people know it so the logistics of moving to safer coding practices and safer hardware are difficult. If transreal arithmetic is adopted by the computing community, things will probably get worse before they get better. It is, therefore, very important that the transition to transreal arithmetic is handled, carefully, by knowledgeable people.

The benefits of improved efficiency and safety highlight the costs of failing to examine and adopt transreal arithmetic.

5. Newton, Euclid and Pythagoras

The present paper is one of a sequence of papers which seeks to develop mathematics so that it allows division by zero in a natural and useful way. One goal of this research is to develop the methods of mathematical physics so that they enable the calculation of physical properties at singularities, as demonstrated in the Tutorial. Much of Newton's work [12] [15] uses methods of real arithmetic, which we have already developed into transreal form, and of calculus, which we have begun to develop [5] [6]. In particular, [5] provides a framework for extending Newton's use of first and ultimate ratios, which are the only explicit appearance of calculus in his *Philosophiae Naturalis Principia Mathematica*. When we examine physical questions using our new mathematical tools, we sometimes accommodate non-finite quantities by explicitly making each parameter of Newton's equations the subject of a defining equation, to which we may add boundary conditions. But we may also look at historical texts to see if there are formulae or methods that apply, without any change, when transnumbers are substituted for whatever other kind of number the historical text assumed. That is, are there any historical formulae or methods that apply lexically to transnumbers?

On examining the Principia, in translation [12] [13], we find that Newton makes use of six classical operations on Euclidean proportions, as set out in the Book 5 of Euclid's *Elements* ([31] vol. 2, especially pp. 114-115). Following the practice in Newton's time, these are given Latin names in [12] p. 313-314, where references are given to some, but not necessarily all, of their occurrences in Newton's Principia. We repeat these references here:

alternando (book 1, prop. 45, ex. 2); *convertendo* (book 1, prop 94, case 1; book 2, lem. 1); *componendo* (book 1, prop. 1; prop. 20, case 2); *dividendo* (book 1, prop. 20, case 1; book 2, prop. 6); *ex aequo* (book 1, prop. 39, corol. 3; prop 71); *ex aequo perturbate* (book 1, lem. 24; book 2, prop. 30). We show that four of these – *componendo, dividendo, convertendo, ex aequo* – apply to any transreal numbers when the Euclidean proportions, written in the modern form, $A : B$, are re-written as proper transreal-fractions, A'/B', of any transreal A' and B'. Here the diacritical prime marks the fact that the numerator, A', and denominator, B', are in least terms. For every Euclidean length, A and B, with $0 < A, B < \infty$, there is some proper transreal-fraction $A'/B' = A/B$ but, in general, $A' \neq A$ and $B' \neq B$ because the A', B' are in least terms while the A, B are general. In particular, these classical operations apply to negative, zero, and non-finite lengths that Euclid did not consider.

During the proofs we sometimes form a reciprocal, so that in passing from A'/B' to B''/A'' we use a second diacritical mark to denote a second reduction to least terms which may, here, carry the sign of a transreal number from the numerator, A', to the numerator, B''. The concatenation of primes is carried out as many times as needed. We find that one operation, *ex aequo perturbate,* branches into two forms, depending on whether the lengths are all finite and non-zero or else some of them are zero or non-finite. If Newton had used these transreal forms of *ex aequo perturbate,* he would have been obliged to break his argument into the two cases of a non-zero, finite solution and a zero or non-finite solution. It seems that Newton uses the Euclidean *ex aequo perturbate* in just one place in his Principia: book 2, prop. 30; page 710 in [12]. This proposition deals with the motion of a pendulum acting against a resisting force and has no influence on Newton's *Book 1: The Motion of Bodies* or *Book 3: The System of the World* that develop, respectively, Newton's dynamical and gravitational theories. We find that the last operation, *alternando,* applies to any transreal numbers when a canonical form of proportions is used prior to casting the proportions into proper transreal-fractions. This operation can be carried out without branching so Newton could have used the transreal form of *alternando* without modifying his argument. It seems that Newton uses the Euclidean *alternando* in just one place in his Principia: book 1, lemma 20, case 1; pp. 485-486 in [12]. But the influence of lemma 20 spreads out from there.

In Euclid's time, neither negative numbers nor zero were known so in Euclidean proportions, written in the modern form, $A : B$, the lengths A and B are positive and finite. That is, $0 < A, B < \infty$. This is so even where a subtraction is involved, as in the length $A = B - C$. The length A is always positive. If $B > C$ then A is marked off in the direction of B; if $B < C$ then A is marked off in the direction of C, which is opposite to the direction of B; but if $B = C$ then A does not exist. There is no zero length in Euclid's proportions. It is as surprising that *componendo, dividendo, convertendo,* and *ex aequo,* as given by Euclid, apply to zero as it is surprising that they apply to infinity. We might ask why these operators were not extended by Newton? Newton recommended the use of negative numbers and zero when working algebraically with geometrical lengths ([15] vol. 5, p. 59). He set out division as multiplication by a reciprocal ([15] vol. 5, p. 81), as both we and Carlström [25] do, but Newton gave no indication, in the Principia, of how the reciprocal of zero should be handled. This prevents the extension of the Euclidean operations on proportions, because both zero and its reciprocal, infinity, are needed to preserve symmetry within the proportions. While Newton understood zero, and all of rational arithmetic, as evidenced by his *Arithmetica Universalis* ([15] vol. 5, especially pp. 52-109), which rational arithmetic is the basis of transreal arithmetic, and understood his own, and Leibniz's, calculus of limits, and understood infinity as an unbounded number, yet he considered an exact infinity to be paradoxical ([15] vol. 1, pp. 89-90). Still less, did Newton understand nullity, which is needed to make arithmetical operations total. Newton was also blocked by his understanding of equality. In modern notation, Newton held that $a = b \Leftrightarrow a - b = 0$. (See [15] vol. 5, especially pp. 110-113.) This prevented Newton from recognising that non-finite a, b can be equal, because, in this circumstance, $a - b = \Phi \neq 0$. Quite simply, Newton had no access to transreal arithmetic which would have allowed him to generalise Euclid's operations on proportions to non-finite numbers, and which would have allowed him to generalise both his arithmetic and calculus. It is not until a set theoretical notion of equality is available, that it is possible to do these things.

Euclid's writings date to the 3rd Century B.C., but they are, in large part, a compendium of earlier knowledge. Operations on proportions of numbers, treated in book 7 of the Elements ([31] vol 2), are said, ([31] pp 112-113), to date to Pythagoras [6th Century B.C.] who introduced the harmonic mean from the Babylonians. The Babylonians must, therefore, have known this proportion at an earlier date. Thus, some of the mathematical methods in use in the 6th Century B.C., and earlier, could be extended, by the modern mathematician, to allow division by zero.

Our intention is not to examine Newton's work in its historical context [32], nor to interpret it in modern form [33], but to show that the Euclidean and Pythagorean mathematics used by Newton can be extended to deal with transreal numbers. We invite the reader to consider how naturally transreal arithmetic extends these early notions of proportions? The first four operations on proportions are extended lexically, the latter two involve the introduction of a computational path that would have been alien to historical writers.

Note, very carefully, that the following proofs are carried out in transreal arithmetic, not in real arithmetic. However, a reader who has not learned transreal arithmetic may read the first four proofs, lexically, as real proofs. That is, these four transreal proofs apply, without change, to real numbers. The remaining two proofs cannot be read as real proofs. The reader, who wishes to follow them, has no option but to learn transreal arithmetic.

5.1 Componendo

It is to be proved that if $A : B = C : D$ then $A + B : B = C + D : D$ where A, B, C, D are arbitrary transreal numbers.

$$\text{Let } \frac{A'}{B'} = \frac{C'}{D'} \text{ then} \tag{20}$$

$$\frac{A'}{B'} + \frac{1}{1} = \frac{C'}{D'} + \frac{1}{1} \Rightarrow \tag{21}$$

$$\frac{A' \times 1 + B' \times 1}{B' \times 1} = \frac{C' \times 1 + D' \times 1}{D' \times 1} \Rightarrow \tag{22}$$

$$\frac{A' + B'}{B'} = \frac{C' + D'}{D'} \text{ Q.E.D.} \tag{23}$$

5.2 Dividendo

It is to be proved that if $A : B = C : D$ then $A - B : B = C - D : D$ where A, B, C, D are arbitrary transreal numbers. The proof is similar to the above, but we give it explicitly.

$$\text{Let } \frac{A'}{B'} = \frac{C'}{D'} \text{ then} \tag{24}$$

$$\frac{A'}{B'} - \frac{1}{1} = \frac{C'}{D'} - \frac{1}{1} \Rightarrow \tag{25}$$

$$\frac{A' \times 1 - B' \times 1}{B' \times 1} = \frac{C' \times 1 - D' \times 1}{D' \times 1} \Rightarrow \tag{26}$$

$$\frac{A' - B'}{B'} = \frac{C' - D'}{D'} \text{ Q.E.D.} \tag{27}$$

5.3 Convertendo

It is to be proved that if $A : B = C : D$ then $A : A - B = C : C - D$ where A, B, C, D are arbitrary transreal numbers.

$$\text{Let } \frac{A'}{B'} = \frac{C'}{D'} \text{ then} \tag{28}$$

$$\frac{B''}{A''} = \frac{D''}{C''} \Rightarrow \tag{29}$$

$$\frac{-B''}{A''} = \frac{-D''}{C''} \Rightarrow \tag{30}$$

$$\frac{1}{1} + \frac{-B''}{A''} = \frac{1}{1} + \frac{-D''}{C''} \Rightarrow \tag{31}$$

$$\frac{1 \times A'' + 1 \times (-B'')}{1 \times A''} = \frac{1 \times C'' + 1 \times (-D'')}{1 \times C''} \Rightarrow \tag{32}$$

$$\frac{A'' - B''}{A''} = \frac{C'' - D''}{C''} \Rightarrow \tag{33}$$

$$\frac{A'''}{A''' - B'''} = \frac{C'''}{C''' - D'''} \text{ Q.E.D.} \tag{34}$$

5.4 Ex aequo

It is to be proved that if $A : B = C : D$ and $E : F = G : H$ then $A \times E : B \times F = C \times G : D \times H$ where A, B, C, D, E, F, G, H are arbitrary transreal numbers.

$$\text{Let } \frac{A'}{B'} = \frac{C'}{D'} \text{ and } \frac{E'}{F'} = \frac{G'}{H'} \text{ then} \tag{35}$$

$$\frac{A'}{B'} \times \frac{E'}{F'} = \frac{C'}{D'} \times \frac{G'}{H'} \Rightarrow \tag{36}$$

$$\frac{A' \times E'}{B' \times F'} = \frac{C' \times G'}{D' \times H'} \text{ Q.E.D.} \tag{37}$$

5.5 Ex aequo perturbate

It is to be proved, equation (41), that if $A : B = F : G$ and $B : C = E : F$ then $A : C = E : G$ where A, B, C, E, F, G are non-zero, real numbers. In a departure from Euclid, it is also to be shown, equation (40), that $A \times B : B \times C = E \times F : F \times G$ for all transreal numbers.

$$\text{Let } \frac{A'}{B'} = \frac{F'}{G'} \text{ and } \frac{B'}{C'} = \frac{E'}{F'} \text{ then} \tag{38}$$

$$\frac{A'}{B'} \times \frac{B'}{C'} = \frac{E'}{F'} \times \frac{F'}{G'} \Rightarrow \tag{39}$$

$$\frac{A' \times B'}{B' \times C'} = \frac{E' \times F'}{F' \times G'} \Rightarrow \quad \text{Q.E.F.} \tag{40}$$

$$\frac{A'}{C'} = \frac{E'}{G'} \text{ by cancellation of real, non-zero } B', F'. \text{ Q.E.F.} \tag{41}$$

5.6 Alternando

It is to be proved that if $A : B = C : D$, in a particular canonical form, then $A : C = B : D$ where A, B, C, D are arbitrary, transreal numbers.

We begin by re-writing zero and infinite proportions in a canonical form. Other proportions need not be re-written. This manoeuvre preserves sign information in the product of fractions.

$$\begin{aligned}
&\text{if } \frac{A}{B} = 0 \text{ then } 0 \to A,\, 1 \to B,\, 1 \to C,\, \infty \to D \text{ endif} \\
&\text{if } \frac{A}{B} = \infty \text{ then } \infty \to A,\, 1 \to B,\, 1 \to C,\, 0 \to D \text{ endif} \\
&\text{if } \frac{A}{B} = -\infty \text{ then } \infty \to A,\, -1 \to B,\, -1 \to C,\, 0 \to D \text{ endif}
\end{aligned} \tag{42}$$

$$\text{Let } \frac{A}{B} = 0 \text{ then } \frac{C}{D} = 0 \text{ and } \frac{A}{B} \times \frac{B}{C} = \frac{C}{D} \times \frac{B}{C} \Rightarrow \frac{0}{1} \times \frac{1}{1} = \frac{1}{\infty} \times \frac{1}{1} \Rightarrow \frac{0}{1} = \frac{1}{\infty} \Rightarrow 0 = 0 \text{ Q.E.F.} \tag{43}$$

Let $\frac{A}{B} = \infty$ then $\frac{C}{D} = \infty$ and $\frac{A}{B} \times \frac{B}{C} = \frac{C}{D} \times \frac{B}{C} \Rightarrow \frac{\infty}{1} \times \frac{1}{1} = \frac{1}{0} \times \frac{1}{1} \Rightarrow \frac{\infty}{1} = \frac{1}{0} \Rightarrow \infty = \infty$ Q.E.F. (44)

Let $\frac{A}{B} = -\infty$ then $\frac{C}{D} = -\infty$ and

$$\frac{A}{B} \times \frac{B}{C} = \frac{C}{D} \times \frac{B}{C} \Rightarrow \frac{\infty}{-1} \times \frac{-1}{-1} = \frac{-1}{0} \times \frac{-1}{-1} \Rightarrow \frac{-\infty}{1} \times \frac{1}{1} = \frac{-1}{0} \times \frac{1}{1} \Rightarrow \frac{-\infty}{1} = \frac{-1}{0} \Rightarrow -\infty = -\infty \text{ Q.E.F.} \quad (45)$$

Let $\frac{A}{B} \in R \setminus \{0\}$ then $\frac{C}{D} \in R \setminus \{0\}$ and $B, C \in R \setminus \{0\}$, whence

$\frac{A}{B} \times \frac{B}{C} = \frac{C}{D} \times \frac{B}{C} \Rightarrow \frac{A'}{C'} = \frac{B'}{D'}$ by cancellation of real, non-zero B, C. Q.E.D. (46)

5.7 Discussion

Let us be clear what the above argument shows. We maintain that Newton's arithmetic is extended by transreal arithmetic [1], that the minimal use he makes of explicit appeals to calculus is extended by transreal topology [5], and that the methods of proportions he uses are extended to transreal ratios, as just proved. Thus, all of the mathematics in Newton's Principia is extended to transreal numbers. It would be possible, therefore, to re-write the Principia so that all of the physics in it applies at singularities. While this might be of historical interest, there are more pressing matters at hand, amongst them how to apply transnumbers to modern physics, which is largely stated in complex numbers.

6. Transcomplex Numbers

It is possible to design mathematical structures and operations on them in exactly the same way as data structures and programs are designed. This has the practical effect that a very large number of programmers might become creative mathematicians. Let us see how a programmer might partially extend complex numbers to transcomplex numbers. We then show that all algebraic approaches, based on generalising Cartesian-complex arithmetic, fail; but that a geometrical approach to constructing transcomplex numbers succeeds.

The programmer encodes complex numbers as a tuple, (r, θ), of a radius, r, and an angle, θ, and implements the operations of complex multiplication, division, addition, and subtraction. The programmer then loads subroutines that implement transreal arithmetic and finds that the complex multiplication and division subroutines now work on any transreal r and θ, including zero and non-finite numbers. In particular, the programmer can compute $(1, \theta) \div (0, 0) = (\infty, \theta)$, thereby computing any particular point at infinite radius and finite, that is real, angle. Such points are not distinguished in complex analysis, where all points at infinite distance are taken to be equivalent, regardless of angle. Thus, the programmer has created a new mathematics. The programmer also finds that addition and subtraction now work for any transreal arguments. But there are degenerate cases, such as $(\infty, 0) + (\infty, \pi/2) = (\Phi, \Phi)$, where the programmer might reasonably have expected the parallelogram rule, implemented in the addition subroutine, to compute a point at infinite radius and intermediate angle: $(\infty, 0) + (\infty, \pi/2) = (\infty, \pi/4)$. On tracing the code, the programmer finds that the x-component of the parallelogram sum is computed as

$x = \infty\cos(0) + \infty\cos(\pi/2) = \infty \times 1 + \infty \times 0 = \infty + \Phi = \Phi$, and, similarly, for the y-component. The programmer has discovered, empirically, that in polar-complex form, multiplication and division generalise well, but addition and subtraction, defined via trigonometrical power series, do not.

Complex numbers are ordinarily defined in terms of Cartesian components and formal operations on them; but replacing the real-numbered components with transreal components fails for multiplication and division. For example, on writing $(a + ib)(c + id) = a(c + id) + ib(c + id)$, we note that transreal arithmetic is only partially distributive [1] so we cannot multiply out the factors to obtain a formal definition of multiplication. Similarly, division fails. Addition and subtraction can be defined formally, but they are degenerate. For example, in polar form we expect $(\infty, 0) + (\infty, \pi/2) = (\infty, \pi/4)$, but on writing this in Cartesian form we obtain $(\infty + i0) + (\infty + i\infty) = (\infty + i\infty)$, which holds for any polar (∞, θ) with $0 < \theta < \pi/2$. In other words, the angle is indeterminate within a quarter rotation. For these reasons, there is no algebraic extension of Cartesian-complex numbers to transcomplex numbers. If we are to succeed, we must find another method for extending real numbers to complex numbers, and this new method must also carry transreal numbers into transcomplex ones.

We lay out the non-negative part of the real number line and sweep it through a full rotation about zero. The swept surface is the complex plane. We define multiplication as the usual composition of a dilatation and a clockwise rotation about zero. Similarly, we define division as the usual dilation and an anti-clockwise rotation about zero. Finally, we define addition and subtraction by the parallelogram rule, where the parallelogram is constructed by producing lines geometrically. In other words, we do not specify a computation of the parallelogram in terms of sine and cosine power series. Thus, we give a geometrical construction of the complex numbers and their elementary arithmetical operations. This construction immediately generalises so that the extended-real numbers generate the complex numbers, extended by a circle at infinity, but we have a choice about how to sweep transreal nullity. We must place nullity off our extended-complex plane so that its position is consistent with transreal topology [5]. If we place nullity on an axle, orthogonal to the complex plane and passing through zero, then the rotation which generates the extended-complex plane carries the single point Φ onto itself; but if we place nullity anywhere else then the rotation sweeps out a circle of points (Φ, θ). In our exploration of transcomplex mathematics we have found no convincing use for a circle at nullity so we adopt the former choice of setting transreal nullity above zero. We will revise this decision if any convincing use for a circle at nullity is found. In the mean time, we accept that the lexical circle at nullity, (Φ, θ), is equivalent to the single swept nullity. We must now choose whether to label the single swept nullity as $(\Phi, 0)$ or (Φ, Φ). On examining the Riemann sphere, which we do later, we find that it is convenient to project the north pole of the sphere onto the point (∞, Φ) so, for completeness, we add the transreal axle, (r, Φ), orthogonal to the complex plane and passing through zero. This axle gives rise to a double cover $(0, 0)$ and $(0, \Phi)$ at the origin of the complex plane – which covers we agree to hold topologically distinct. This axle also passes through (Φ, Φ) so we choose this as the label for the swept nullity, recalling that we have already accepted the equivalence $(\Phi, \theta) \equiv (\Phi, \Phi)$ for all transreal θ. Similarly, we adopt the equivalence $(0, \theta) \equiv (0, 0)$ for all real θ. Thus, we have set out a geometrical construction of elementary arithmetic on our extended-complex plane, and the axle at angle nullity; and we have given every point in the construction a unique label, when we respect the double cover at zero and the equivalence classes on zero and nullity. This configuration is shown below. The axle at angle nullity is shown on the z-axis. Our extended-complex plane, (x, y), with a circle at infinity, is swept out by an extended-real radius rotated by a real angle θ. (We note, in passing, that defining transcomplex arithmetic in terms of geometry, reprises the role of Euclidean proportions in the development of real arithmetic.)

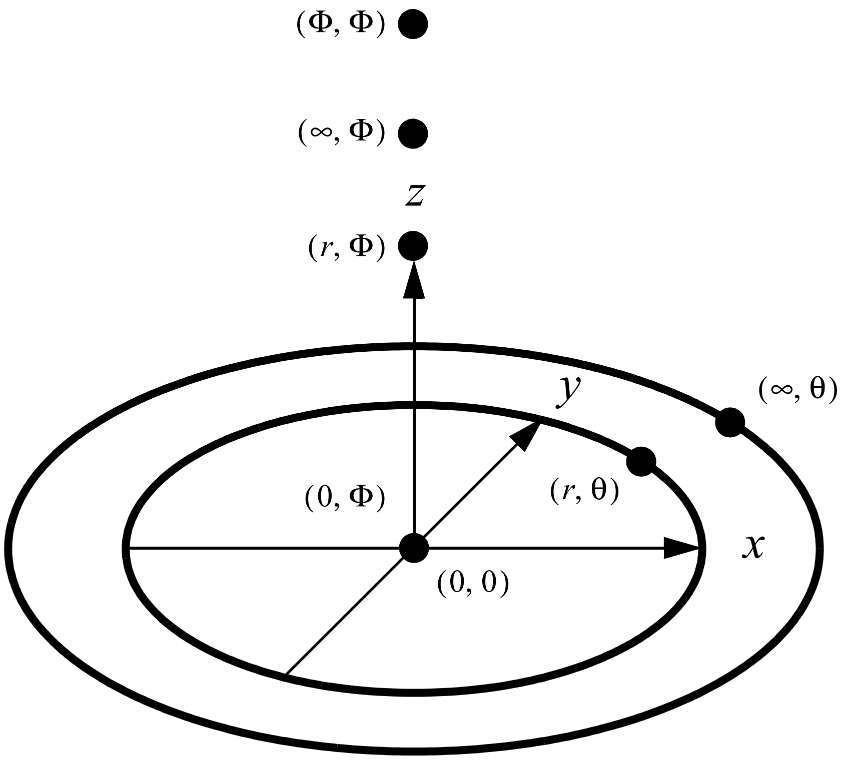


Figure 4: The axle, (z), at angle nullity; and our extended-complex plane, (x, y), swept out by an extended-real radius, rotated by a real angle, theta.

Notice that the decision to sweep the transreal number-line, through a single rotation, restricts multiplication and division to operating on the argument of the angles involved. This is the usual constraint and is consistent with addition and subtraction, because parallelogram rules also operates on the argument of angles. Thus, none of our arithmetical operations respect a Riemann surface that is taken through many rotations. Consequently, we are obliged to accept a cut somewhere in the complex plane where large angles meet small ones, say at $\theta_1 = \pi$ and $\theta_2 = -\pi$. We loose continuity at this cut. However, if we define transcomplex numbers as a three-tuple, (r, c, s), with $c = \cos\theta$ and $s = \sin\theta$ then, when r, c, s are all real, the tuple is a concatenation of three continuous parameters and is, itself, continuous. Hence, we have no use for Riemann surfaces, nor does the complex plane, being the finite part of both our and the ordinary extended-complex plane, contain any cuts. However, depending upon the particular transcomplex function in hand, we may wish to introduce cuts in the ordinary complex domain, that is in the finite part of the transcomplex domain, of the function and, in the transcomplex case, we might wish to introduce non-finite cuts to separate the infinite or nullity parts from the finite part. Thus the use of three-tuples improves the continuity of both ordinary complex and transcomplex functions.

The use of a three-tuple brings an immediate practical advantage to computation and might bring a theoretical advantage to mathematical physics. Firstly, digital computation on c, s is much faster than evaluation of the sine, cosine, and arctangent power series. We develop this efficient three-tuple scheme after examining the Riemann sphere, and give an implementation of it as an on-line appendix. Secondly, the tuples (r, c, s) have a minimum modulus of unity, because the minimum modulus of r is zero and (c, s) has a constant, minimum modulus of unity. In both cases the moduli may be nullity, but nullity is neither a minimum nor a maximum. Consequently, physical functions that are currently thought to operate as $1/r$, where r is the modulus of a complex number, might, instead, operate as $1/(r+1)$, in some units, where the term $r+1$ is the modulus of a three-tuple. In other words, the physical universe might be better described in three-tuple space than in complex space. If so, the three-tuple ensures, by hypothesis, that physical functions are defined everywhere in a finite domain, without cuts and, as a consequence of the modulus $r+1$, have no infinities or nullities arising from a zero radius. Such functions would flatten out as r becomes small. It is extremely unlikely that empirical physicists have missed such an effect; but it is an empirically testable prediction which, if confirmed, would provide a very powerful argument for adopting three-tuple arithmetic. In this paper, however, we develop a mathematics which does allow finite, infinite and nullity behaviour at singularities so that theoretical physicists, and others, have the opportunity to use a total mathematics that can be computed efficiently, without resorting to special mathematical functions at singularities.

7. Riemann Sphere

The Riemann sphere is widely used in complex analysis and mathematical physics. Its properties are, therefore, of both theoretical and practical interest. We find that transreal arithmetic preserves all of the properties of the ordinary Riemann sphere and provides some new properties at infinity and nullity, potentially opening up new opportunities for research in mathematics and physics.

The Riemann sphere takes part in the stereographic projection of the complex plane. This projection bijectively maps each point on the plane to each point on the Riemann sphere, except for the point at the 'north' pole which is mapped to a non-complex point called *complex infinity*. Traditionally, the sphere lies on the plane with its 'south' pole in contact with the plane ([10], fourth lecture, pp 131-161), but the complex plane can be taken anywhere parallel to this plane and is sometimes taken in the equator ([9] p. 45). We now give a longish quotation from a text written in 1972 ([8] pp. 204-206). This explains the Riemann sphere and is unusually frank in discussing the inability of contemporary mathematics to divide by zero. This places our development of the Riemann sphere into its proper mathematical setting and serves as an illustration of the mathematical importance of transreal and transcomplex arithmetic. In the quotation, we have added equation numbers so that we can conveniently refer to equations. We have mapped the existing equation numbers and figure number so that they fit in with the numbering in this paper. We have re-drawn the figure and have replaced the symbol Φ by Π to denote a projection. We have given a footnote in line, without indicating the footnote. We have slightly modified the layout. We have corrected an italicised S to an emboldened $\boldsymbol{S}$. We have set italic emphasis as emboldened emphasis and have set the whole of the quotation in italic.

We will examine other special functions as we proceed, but first we want to discuss the behaviour of a function "at infinity." There is no complex number that plays the role of this hypothetical point "infinity;" nevertheless, in various situations it can be very helpful, and enlightening, to think of this number as if it really existed. We shall do this by taking the complex plane $\boldsymbol{C}$ *and adjoining to it a new point, labelled* ∞, *to form the extended complex plane* $\boldsymbol{C}^* = \boldsymbol{C} \cup \{\infty\}$. *The new point* ∞ *will be referred to as the* ***point at infinity.***

We might entertain the idea of defining the usual algebraic operations, sum, product, and quotient, in this enlarged system $\boldsymbol{C}^*$, *and this can be done to a certain extent. Naturally, we take the usual operations for pairs of numbers in the subset* $\boldsymbol{C} \subseteq \boldsymbol{C}^*$, *the* ***ordinary complex numbers*** *in* $\boldsymbol{C}^*$. *Our intuition also tells us how we should define* ***some*** *of the operations between ordinary complex numbers and the exceptional point* ∞*; the following rules are almost self-evident:*

$$\begin{aligned} \infty \cdot z &= \infty = z \cdot \infty && \text{for } z \neq 0 \text{ in } \boldsymbol{C} \\ \infty \pm z &= \infty = z \pm \infty && \text{for all } z \text{ in } \boldsymbol{C} \\ z/0 &= \infty && \text{for } z \neq 0 \text{ in } \boldsymbol{C} \\ z/\infty &= 0 && \text{for all } z \text{ in } \boldsymbol{C} \\ \infty \cdot \infty &= \infty \end{aligned} \tag{47}$$

Unfortunately, there is no reasonable way to define all of the familiar operations in the extended number system $\boldsymbol{C}^*$, *due to the fact that* ∞ *does not really behave like an ordinary complex number. In particular, there is no reasonable value we can assign to the combinations*

$$\infty \pm \infty \qquad \infty/\infty \qquad 0/0 \qquad \infty \cdot 0 \qquad 0 \cdot \infty \tag{48}$$

Even in Calculus these are indeterminate forms, and must be left undefined. Notice that $0/\infty = 0$ *and* $\infty/0 = \infty$ *are not indeterminate forms according to the rules we have set up. The combination* $\infty + \infty$ *is indeterminate (and* ***not*** $= \infty$*) since our notion of infinity cannot distinguish between* $+\infty$ *and* $-\infty$*; thus* $\infty + \infty$ *is no better than the obviously indeterminate expression* $\infty - \infty$.

In this book our real desire is to use the extended number system to understand ***geometric*** *problems. For this purpose we will now set up a simple geometric model of the extended complex number system* $\boldsymbol{C}^*$ *in which the exceptional point* ∞ *appears as an actual point. A natural model is provided by the* ***stereographic projection*** *of the* *complex plane onto a sphere. Let us start with three dimensional Euclidean space* $\boldsymbol{R}^3$ *with coordinates* (ξ, η, ζ). *The plane determined by setting* $\zeta = 0$ *is identified with the complex plane by letting* $z = x + iy$ *correspond to the point* $(x, y, 0)$ *so that* $\xi = x$, $\eta = y$, $\zeta = 0$. *Let* N *stand for the point* $(0, 0, 1)$, *which can be regarded as the north pole of the sphere* $\boldsymbol{S}$ *given by the equation*

$$\xi^2 + \eta^2 + (\zeta - 1/2)^2 = (1/2)^2$$

This sphere $\boldsymbol{S}$ *has radius* $r = 1/2$, *and its south pole* $O = (0, 0, 0)$ *rests on the origin of the complex plane. The stereographic projection maps a point* $Z = (x, y, 0)$ *in the complex plane to a point on* $\boldsymbol{S}$ *along the straight line segment that connects* Z *to the polar point* N, *as shown in Figure 5. We write* $\Pi(Z)$ *for the projected point, and when we identify the point* $Z = (x, y, 0)$ *with the corresponding complex number* $z = x + iy$ *in* $\boldsymbol{C}$, *we may regard* Π *as a mapping from* $\boldsymbol{C}$ *into the sphere* $\boldsymbol{S}$, $\Pi : \boldsymbol{C} \to \boldsymbol{S}$.

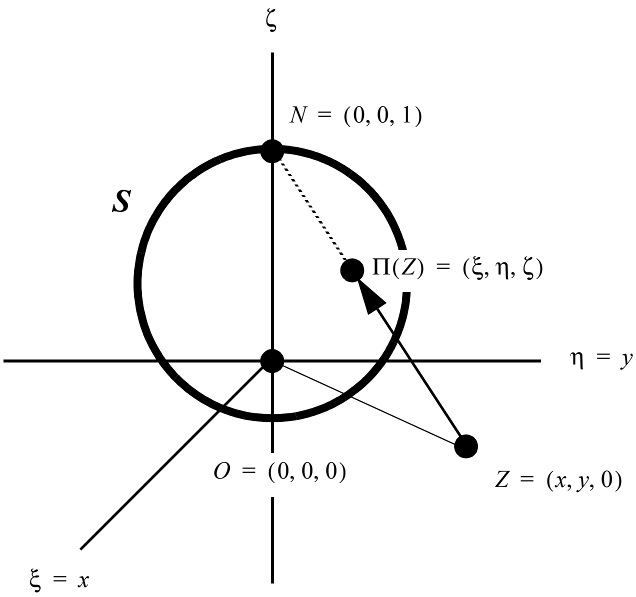


Figure 5: Stereographic projection of the plane $\zeta = 0$ onto the sphere $\boldsymbol{S}$.

Each point $z = x + iy$ *in the complex plane maps to a unique point on* $\boldsymbol{S}$, *and by direct calculations with similar triangles (which we leave to the reader) we see that the projected point has coordinates* $\Pi(Z) = (\xi, \eta, \zeta)$ *given by*

$$\xi = \frac{x}{1+r^2} = \frac{\mathrm{Re}(z)}{1+|z|^2} \qquad \eta = \frac{y}{1+r^2} = \frac{\mathrm{Im}(z)}{1+|z|^2}$$
$$\zeta = \frac{r^2}{1+r^2} = \frac{|z|^2}{1+|z|^2} \qquad (49)$$

where $r^2 = x^2 + y^2$. Conversely, a point (ξ, η, ζ) on the sphere, ***other than the exceptional point*** $\boldsymbol{N = (0, 0, 1)}$*, corresponds to the point $z = x + iy$ in the complex plane such that*

$$x = \frac{\xi}{1-\zeta} \qquad y = \frac{\eta}{1-\zeta} \qquad |z|^2 = \frac{\zeta}{1-\zeta} \qquad (50)$$

It is important to notice that no point in the complex plane projects to the polar point N, while every point in the "punctured sphere" $\boldsymbol{S} \sim \{N\}$ corresponds to a unique complex number. The exceptional role of the polar point N allows us to use the sphere $\boldsymbol{S}$ as a model for the extended complex plane $\boldsymbol{C}^* = \boldsymbol{C} \cup \{\infty\}$, in which the exceptional point ∞ in $\boldsymbol{C}^*$ is identified with N and an ordinary complex number z with its projection $\Pi(z)$. Because we have chosen to represent $\boldsymbol{C}^*$ as a sphere, the extended complex number system $\boldsymbol{C}^*$ is often referred to as the ***complex sphere*** *(or* ***Riemann sphere****).*

Frederick Greenleaf ([8] pp. 204-206)

In the second paragraph, Greenleaf expresses the desire to enlarge complex arithmetic so that it contains ordinary complex arithmetic and applies to infinity. He states that the equations (47) are intuitive. These are equations of both transreal [1] and transcomplex arithmetic so Greenleaf provides literal proof that these parts of transarithmetic are intuitive, as is the idea of providing all the operations of the ordinary arithmetic within an enlarged arithmetic. Greenleaf goes on to state that there is no reasonable way to assign a meaning to the operations in Equation (48) because ∞ does not behave like an ordinary complex number. This is an appeal to ignorance, and was reasonable in the age before division by zero. Today, all of these operations are defined in both transreal and transcomplex arithmetic. We maintain that ∞ is a transcomplex number but, in this section, we elucidate various roles of complex infinity which the Riemann sphere conflates.

Notice that Greenleaf falsely states that $\infty/0 = \infty$ is a determinate form according to the rules he has set up. For this form to be determinate, Greenleaf would have to have $z/0 = \infty$ for all $z \neq 0$ in $\boldsymbol{C}^*$. That is, he would have to replace $\boldsymbol{C}$ by $\boldsymbol{C}^*$. To be Charitable, this is a simple slip, and we allow that Greenleaf agrees with us on the well formed nature of $\infty/0 = \infty$. Greenleaf treats ∞ as an unsigned object. That is, complex ∞ is a *projective infinity* in contrast to the signed, *affine infinities*, $+\infty$ and $-\infty$, in transreal arithmetic. This different treatment of the sign of infinity prevents an ordinary enlargement of complex arithmetic from containing an ordinary enlargement of real arithmetic. By contrast, transcomplex arithmetic provides, firstly, signed infinities, (∞, c, s), at a real angle defined uniquely by c, s, and provides, secondly, an unsigned infinity, (∞, Φ, Φ), at the angle nullity. We show, below, that Greenleaf's ∞ behaves like (∞, Φ, Φ). Thus, we have no difficulty in taking transreal arithmetic as a proper subset of transcomplex arithmetic.

The equations (49) do not have the highest information content when any consistent selection of x, y, r has infinite r. We give these equations more information by splitting the equations into two computational paths, as the parameter is or is not infinite, and by lexically cancelling infinite terms prior to establishing the simplified equations as a definition. Thus, when $|x| = r \neq 0, \infty$ we simplify the equation for ξ as follows:

$$\xi = \frac{x}{1+r^2} = \frac{x}{1+x^2} = \frac{\frac{x}{x}}{\frac{1}{x}+\frac{x^2}{x}} = \frac{1}{\frac{1}{x}+x} \tag{51}$$

Now, when $x = \pm\infty$ we continue the simplification as:

$$\xi = \frac{1}{\frac{1}{x}+x} = \frac{1}{\frac{1}{\pm\infty}+(\pm\infty)} = \frac{1}{0+(\pm\infty)} = \frac{1}{\pm\infty} = 0 \tag{52}$$

The equation for η is obtained similarly. The equation for ζ is obtained as:

$$\zeta = \frac{r^2}{1+r^2} = \frac{\frac{r^2}{r^2}}{\frac{1}{r^2}+\frac{r^2}{r^2}} = \frac{1}{\frac{1}{r^2}+1} \tag{53}$$

and when $x = \pm\infty$ we have

$$\zeta = \frac{1}{\frac{1}{r^2}+1} = \frac{1}{\frac{1}{\infty^2}+1} = \frac{1}{\frac{1}{\infty}+1} = \frac{1}{0+1} = \frac{1}{1} = 1 \tag{54}$$

Note, very carefully, that we have searched for and found computational paths with a high information content. These particular paths give finite results for infinite parameters. The search for such paths currently requires mathematical skill. There is, at present, no known algorithm which reduces a wide class of expressions to highest information content, though the class of fractional linear transformations can be reduced algorithmically ([8] p. 221). Put another way, when we set out a total set of equations in a transarithmetic we must sometimes supply boundary conditions for the non-finite parameters. Using the boundary conditions we have just computed gives:

$$\xi = \begin{cases} 0 & : x = \infty \\ \dfrac{x}{1+r^2} & : \text{otherwise} \end{cases} \qquad \eta = \begin{cases} 0 & : y = \infty \\ \dfrac{y}{1+r^2} & : \text{otherwise} \end{cases} \qquad \zeta = \begin{cases} 1 & : r = \infty \\ \dfrac{r^2}{1+r^2} & : \text{otherwise} \end{cases} \tag{55}$$

Now, for any consistent choice of x, y and r, with r infinite, we have $(\xi, \eta, \zeta) = (0, 0, 1)$, which is the north pole, N, as Greenleaf requires. Similarly, for

$x = y = r = 0$ we have $(\xi, \eta, \zeta) = (0, 0, 0)$, which is the south pole, as Greenleaf requires. But Greenleaf does not consider the case of any consistent choice of x, y and r, with r nullity, which leads to $(\xi, \eta, \zeta) = (\Phi, \Phi, \Phi)$. Thus, we have computed all of Greenleaf's equations (49) using transreal arithmetic to evaluate a set of computational paths which classifies all possible transreal solutions. In other words, we have put Greenleaf's intuitive equations (47) on the firm axiomatic basis of transreal arithmetic [1], and we have used this arithmetic to solve Greenleaf's equations for the projection of the whole of the ordinary extended-complex plane onto the whole of the Riemann sphere.

Greenleaf also gives equations for the projection of a point on the Riemann sphere onto the ordinary extended-complex plane, Equation (50). We now compute these projections using transreal arithmetic, taking care to distinguish Cartesian three-tuples (ξ, η, ζ) from transcomplex three-tuples (r, c, s).

We take it as given that the generality of the Riemann sphere projects uniquely onto the complex plane. We now compute the projection at the two poles. At the south pole, with Cartesian co-ordinates $(0, 0, 0)$, we have $(0, 0, 0) \to z$ with $z = 0 + i0$ and radius $r = 0 = |z|$, because:

$$
\begin{aligned}
x &= \frac{\xi}{1-\zeta} = \frac{0}{1-0} = \frac{0}{1} = 0 \\
y &= \frac{\eta}{1-\zeta} = \frac{0}{1-0} = \frac{0}{1} = 0 \\
|z|^2 &= \frac{\zeta}{1-\zeta} = \frac{0}{1-0} = \frac{0}{1} = 0
\end{aligned}
\tag{56}
$$

This implies that the transcomplex co-ordinates of z are $(0, 1, 0)$, corresponding to a radius of zero at angle zero.

At the north pole, with Cartesian co-ordinates $(0, 0, 1)$, we assert that $(0, 0, 1) \to z$ and compute $r = \infty$, despite the fact that there is no complex z in this case:

$$
\begin{aligned}
x &= \frac{\xi}{1-\zeta} = \frac{0}{1-1} = \frac{0}{0} = \Phi \\
y &= \frac{\eta}{1-\zeta} = \frac{0}{1-1} = \frac{0}{0} = \Phi \\
|z|^2 &= \frac{\zeta}{1-\zeta} = \frac{1}{1-1} = \frac{1}{0} = \infty
\end{aligned}
\tag{57}
$$

The terms $x = \Phi$, $y = \Phi$ give rise to $\theta = \text{arctan2}(x, y) = \text{arctan2}(\Phi, \Phi) = \Phi$, compare with [6]. This solution is also given by $c = x/(\sqrt{x^2 + y^2}) = \Phi/(\sqrt{\Phi^2 + \Phi^2}) = \Phi$, and similarly for s. Hence, an attempt to write z as a complex number fails, because the putative $z = \Phi + i\Phi$ encodes no information about the known radius $r = \infty$. By contrast, z may be written as a transcomplex three-tuple, $z = (\infty, \Phi, \Phi)$, which encodes all of the information about this point.

This is an extremely important result. It is already known that the equations governing the projection of the Riemann sphere cannot be solved at the north pole using real arithmetic. In other words, it is already known that algebraic geometry breaks down in this case. But these same equations do give a solution when we use transreal arithmetic. Thus, transreal arithmetic extends algebraic geometry so that it obtains a solution in at least this one, singular, configuration. We expect that all singular configurations in algebraic geometry can be solved using transreal arithmetic, because this arithmetic is total; but this question can be settled only by further research.

Thus far, we have shown that Greenleaf's account of the Riemann sphere holds everywhere on the sphere when transreal arithmetic is used in place of real arithmetic, and his extension of it, Equation (47), and where the ordinary complex infinity is replaced by transcomplex (∞, Φ, Φ). We now summarise all projections onto transcomplex numbers in Figure 7, but first we introduce the *wheel at infinity.*

Figure 6, below, shows two wheels. A *wheel* is a circle, or *rim,* together with a point, called the *hub,* which is shown at the centre of the circle. The hub is co-punctal with a point on an *axle.* The *wheel at the north pole* is a finite wheel. Its hub is the north pole, with Cartesian co-ordinates, $(0, 0, 1)$. Its rim contains the points with transcomplex co-ordinates, $(1/2, c, s)$, with $c = \cos\theta$ and $s = \sin\theta$ for finite θ, and has unit diameter. The *wheel at infinity* is a non-finite wheel. Its hub has transcomplex co-ordinates (∞, Φ, Φ), which is a point at infinite distance, and nullity angle, from the origin of the complex plane. Its rim contains the points with transcomplex co-ordinates, (∞, c, s), with $c = \cos\theta$ and $s = \sin\theta$ for finite θ, and has infinite diameter. The two-headed arrow indicates a bijective mapping between the wheel at the north pole and the wheel at infinity.

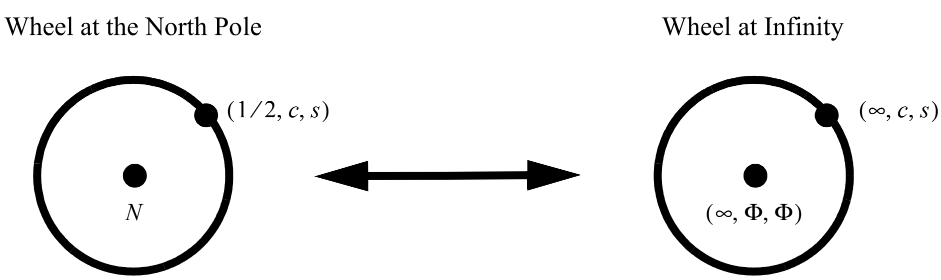


Figure 6: The wheel at the north pole, N, projecting onto the wheel at infinity.

Figure 7, below, shows an elevation of our extended Riemann sphere. The Riemann sphere is shown as a thick circle with the points N, E', S, E lying on its circumference. The thick horizontal line, labelled I', N, I, is a side view of the unit wheel which projects onto the wheel at infinity. The wheel at infinity lies beyond the bounds of the figure, but see Figure 6. The thick vertical line, labelled N, 0, together with the pole star, labelled P, is an axle. It projects onto the axle at angle nullity which may be taken, as a drafting convention, to overlay the axle of the sphere, while remaining topologically distinct from it. The point N is an isolated point. The axle is topologically closed at 0 and open at N, as shown in Figure 4. Correspondingly, the sphere is punctured at N, making the sphere topologically open in a neighbourhood about N, but it is compactified by N. The point at infinite radius and nullity angle, with co-ordinates (∞, Φ, Φ), labelled ∞, is shown as being co-punctal with N. These two points are topologically distinct. The point at nullity, with co-ordinates (Φ, Φ, Φ), labelled Φ, is shown as being co-punctal with P, but is topologically distinct from it.

Drawing ∞ and Φ at these points is a convenient drafting convention, but is not essential to the projections. The point labelled 0 projects to $(0, \Phi, \Phi)$ on the axle at angle nullity. As a drafting convention, it is drawn above the origin of the complex plane and, identically, above the origin of the finite part of the transcomplex plane. The medium horizontal line, passing through e', S, e, is a side view of a central part of the complex plane. The whole plane extends beyond the bounds of the figure. The thin solid lines are projections of finite length. All projections are taken in the direction from N, passing through some, not-necessarily distinct, point to a terminal point. This directionality ensures that the north pole projects onto distinct points on the rim at infinity, and is consistent with both finite and nullity projections – so it is convenient to demand that this directionality holds everywhere. This drafting convention requires that the point at nullity is shown on the axle and below the origin of the complex plane, whereas there is freedom to show it above the origin in Figure 4. The *pole star,* P, is a point at unit distance below the origin. It projects onto the point at nullity, Φ. The thin dotted lines are projections of non-finite length. The projections from N, passing through I and I' to, respectively, i and i' (beyond the bounds of the figure), are projections of infinite length. The projection from N passing through N projects to ∞ and is also of infinite length, despite appearances in the diagram. Finally, the projection from N, passing through any other point on the axle, projects bijectively to a point on the axle at angle nullity, and is of nullity length. As usual, a projection from N through S projects onto the origin of the complex plane, which origin is co-punctal with S.

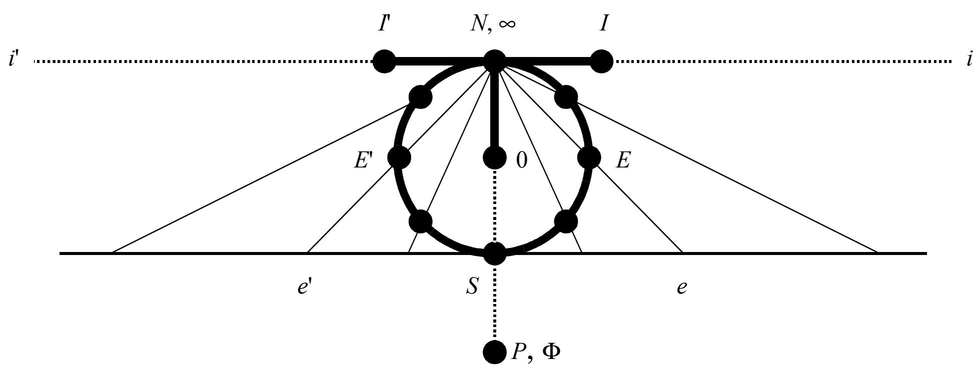


Figure 7: Transcomplex superset of the Riemann sphere and its projections.

Now we see an infelicity in Greenleaf's construction and uncover a new class of transcomplex limits. Greenleaf assumed that the north pole maps onto a non-complex point, ∞, called complex infinity; but, as we have argued, it maps onto transcomplex (∞, Φ, Φ) which is a point at infinite radius and angle nullity. We also have points, (∞, c, s), which are points at infinite radius and finite angle. Now consider a function which grows monotonically in magnitude, without finite bound. If this function does not converge to a finite orientation then we define that its limit is (∞, Φ, Φ), but if it does converge to a finite orientation, θ, with $c = \cos\theta$ and $s = \sin\theta$, then we define that its limit is (∞, c, s).

Greenleaf takes all of these cases as equivalent. Consequently, that part of mathematical physics that is based on complex analysis, ignores nullity limits and conflates all of the angles of infinite limits, whereas we preserve the angle information for all limits, whether finite or non-finite. Consequently, we are better able to take on the analysis of physical singularities.

It should be known to the reader that Greenleaf's presentation of the Riemann sphere is just one of several parameterisations that appear in the mathematical literature. We have quoted extensively from Greenleaf because he is particularly frank in discussing the motivation for his developments. This is a strength in a text book. Different, parameterisations of the Riemann sphere lead us to change the details of our criticism, but we arrive at the same conclusion: transcomplex arithmetic is more capable than complex arithmetic at computing properties at singularities.

8. Transcomplex Arithmetic

Now that we have defined some geometrical properties of the transcomplex numbers and their projections, it remains to define transcomplex arithmetic. We begin by reviewing non-finite angles and distances. We find that there is just one non-finite angle: nullity; but there are two non-finite distances: infinity and nullity. We then give algebraic definitions of the operations of multiplication, division, addition, and subtraction. After which, we give geometrical constructions for these operations. Finally, we give various proofs showing how transcomplex arithmetic relates to other arithmetics.

8.1 Non-Finite Angle

Consider the Taylor series for each and every real trigonometrical series. Such series, $S(x)$, are a function of an angle, x, and have alternating positive and negative terms. But $\infty - \infty = \Phi$ and $-\infty + \infty = \Phi$ and $\Phi - \Phi = \Phi$ and $-\Phi + \Phi = \Phi$ so the sum of the series for $x \in \{-\infty, \infty, \Phi\}$ is Φ. Compare with [6]. But then all non-finite angles give rise to the same non-finite sum, nullity, and we define that the angle nullity is the canonical form of these non-finite angles. In particular, when $x \in \{-\infty, \infty, \Phi\}$, we have $\cos(x) = \sin(x) = \Phi$ and $\arccos(x) = \arcsin(x) = \Phi$. Furthermore, $\arctan2(x, y) = \Phi$ when $x \in \{-\infty, \infty, \Phi\}$ or $y \in \{-\infty, \infty, \Phi\}$. In the body of this paper we always construct cosines and sines using these relationships, but the appendix *transarith.p* gives a more general form of $\arctan2(x, y)$ that is a total function of all transreal x, y. See the procedure *stdrcs* in the appendix.

The above result, obtained from the Taylor series of the real trigonometrical functions, is consistent with results obtained from geometry. For example, consider a right triangle with unit hypotenuse, as used to define the real trigonometrical functions via ratios of the lengths of sides of the triangle. Now dilatate the triangle by zero so that all lengths are zero. Consequently all ratios are nullity. We define that the angle the zero hypotenuse makes with the x-axis is nullity. Consequently, all trigonometrical functions of angle nullity are nullity and the arc-trigonometrical functions of nullity are nullity, as above.

8.2 Non-Finite Distance

The axioms of transmetrics are given in [5]. These differ from the axioms of metrics only in that *greater-than-or-equals* is replaced by *not-less-than*. This substitution admits the distance nullity. Transreal arithmetic admits the distance infinity. For convenience, the axioms defining a transmetric, t, are repeated here:

$$t(a, b) = t(b, a) \tag{58}$$

$$t(a, b) \nless 0 \tag{59}$$

$$t(a, b) = 0 \Leftrightarrow a = b \tag{60}$$

$$t(a, b) + t(b, c) \nless t(a, c) \tag{61}$$

The paper [5] gives the Euclidean transmetric for transreal numbers a, b as follows.

$$t(a, b) = \begin{cases} 0 : a = b \\ \sqrt{(a-b)^2} : \text{otherwise} \end{cases} \tag{62}$$

It should now be understood that when a, b are transcomplex, the Euclidean transmetric, t, is defined on $c = a - b$, with $c = (r_c, c_c, s_c)$, as:

$$t(a, b) = \begin{cases} 0 : a = b \\ \sqrt{r_c^2} : \text{otherwise} \end{cases} \tag{63}$$

Note the simplification $\sqrt{r_c^2} = r_c$.

8.3 Shorthand Polar Form of Transcomplex Numbers

We have defined that a transcomplex number, (r, c, s), has transreal components r, c, s such that $r \nless 0$, and $c = \cos(\theta)$, $s = \sin(\theta)$ for any transreal θ. This is the form in which all transcomplex arithmetic operates. However, once the non-finite angle and distances are understood, it becomes natural to refer to these three-tuples by their polar from as two-tuples (r, θ). Here θ is the argument of an underlying angle, θ_u, given by, for example, $\theta = \text{arctan2}(\cos\theta_u, \sin\theta_u)$ so that $-\pi < \theta \le \pi$ with $\theta + 2k\pi = \theta_u$ for some integer k.

8.4 Definition of the Arithmetical Operators

We define transcomplex multiplication and division so that they contain polar-complex multiplication and division. In fact, the transcomplex definitions are lexically identical to the polar definitions when polar operations on angles are expanded by trigonometric identities to operations on cosines and sines. Addition is implemented via Newton's parallelogram rule with an extension to deal with the sum of a vector at angle nullity with a vector that is not at angle nullity. Note that Newton's rule, [12] pp 111-112 & 417-418, is more general than the modern parallelogram rule, as illustrated in the next section. Subtraction is implemented as addition with the subtrahend being produced by multiplying an addend by a half rotation in the complex plane so as to negate it.

All of the operators have their usual precedence. They are implemented in the on-line appendix *transarith.p*. Geometrical constructions for the arithmetical operators are given in the next section.

$$(r_1, c_1, s_1) \times (r_2, c_2, s_2) = (r_3, c_3, s_3) \text{ where}$$
$$r_3 = r_1 \times r_2$$
$$c_3 = \begin{cases} 1 & : r_3 = 0 \\ c_1 \times c_2 - s_1 \times s_2 & : \text{otherwise} \end{cases} \tag{64}$$
$$s_3 = \begin{cases} 0 & : r_3 = 0 \\ s_1 \times c_2 + c_1 \times s_2 & : \text{otherwise} \end{cases}$$

$$(r_1, c_1, s_1) \div (r_2, c_2, s_2) = (r_3, c_3, s_3) \text{ where}$$
$$r_3 = r_1 \div r_2$$
$$c_3 = \begin{cases} 1 & : r_3 = 0 \\ c_1 \times c_2 + s_1 \times s_2 & : \text{otherwise} \end{cases} \tag{65}$$
$$s_3 = \begin{cases} 0 & : r_3 = 0 \\ s_1 \times c_2 - c_1 \times s_2 & : \text{otherwise} \end{cases}$$

$$(r_1, c_1, s_1) + (r_2, c_2, s_2) = \begin{cases} (r_1 + r_2, \Phi, \Phi) & : \text{any of } c_1, s_1, c_2, s_2 = \Phi \\ (r_3, c_3, s_3) & : \text{otherwise, where} \\ \quad r_1' = \begin{cases} 1 & : r_1 = r_1 + r_2 \\ r_1 \div (r_1 + r_2) & : \text{otherwise} \end{cases} \\ \quad r_2' = \begin{cases} 1 & : r_2 = r_1 + r_2 \\ r_2 \div (r_1 + r_2) & : \text{otherwise} \end{cases} \\ \quad x = r_1' \times c_1 + r_2' \times c_2 \\ \quad y = r_1' \times s_1 + r_2' \times s_2 \\ \quad r_3' = \sqrt{x^2 + y^2} \\ \quad r_3 = r_3' \times (r_1 + r_2) \\ \quad c_3 = \begin{cases} 1 & : r_3' = 0 \\ x \div r_3' & : \text{otherwise} \end{cases} \\ \quad s_3 = \begin{cases} 0 & : r_3' = 0 \\ y \div r_3' & : \text{otherwise} \end{cases} \end{cases} \tag{66}$$

$$(r_1, c_1, s_1) - (r_2, c_2, s_2) = (r_1, c_1, s_1) + (1, -1, 0) \times (r_2, c_2, s_2) \tag{67}$$

8.5 Geometrical Construction of the Arithmetical Operators

Figure 8 shows a Cartesian co-ordinate frame with axes labelled x, y, z. The x- and y-axes have arrow heads, showing the sense of the axis and indicating that the axis extends indefinitely far in both the positive and negative directions. The z-axis has no arrow head, indicating that the whole of the non-negative transreal axis is drawn in the figure. Compare with Figure 1 and Figure 4. A wheel with unit radius is swept along the z-axis. Thus, a polar-transcomplex number, (r, θ), appears on the wheel at height $z = r$, for any non-negative transreal r, and at an angle θ about the z-axis, measured anti-clockwise as viewed from the positive z-axis. However, transcomplex arithmetic is defined with θ in the principal range $-\pi < \theta \le \pi$ or $\theta = \Phi$. A transcomplex number $(r, \theta) \equiv (r, c, s)$ lies on the swept wheel, but the rim at zero, $(0, \theta)$, with θ finite, is identified with transcomplex $(0, 0) \equiv (0, 1, 0)$. The rim at nullity, (Φ, θ), also with θ finite, is identified with $(\Phi, \Phi) \equiv (\Phi, \Phi, \Phi)$. With these identifications, the points on the swept wheel are bijective with the transcomplex numbers and all arithmetical operations on transcomplex numbers are transformations of the swept wheel.

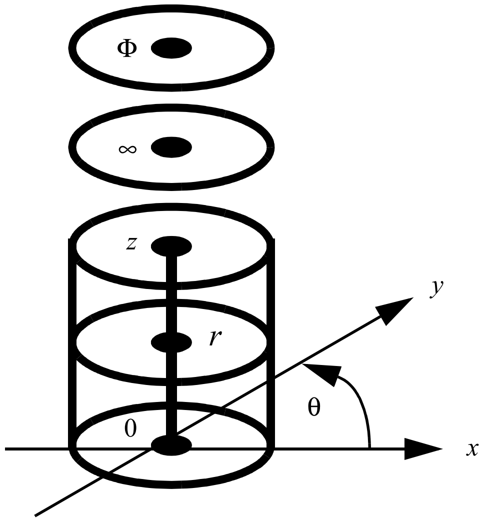


Figure 8: Wheel swept along a non-negative transreal axis.

It will be readily apparent, to the reader, that multiplications and divisions are screws in the swept wheel, that is, they are a combination of a rotation and a translation in the swept wheel. The construction of addition is more involved.

Notice that multiplication and division – Equations (64), (65) – involving any number at angle nullity, produce a resultant in which the phase (angle) collapses to nullity, but the product or quotient of the magnitudes (radii) is maintained. Thus:

$$(r_1, c_1, s_1) \times (r_2, \Phi, \Phi) = (r_1 \times r_2, \Phi, \Phi) \tag{68}$$

$$(r_1, c_1, s_1) \div (r_2, \Phi, \Phi) = (r_1 \div r_2, \Phi, \Phi) \tag{69}$$

By analogy with this behaviour, Equation (66) specifies:

$$(r_1, c_1, s_1) + (r_2, \Phi, \Phi) = (r_1 + r_2, \Phi, \Phi) \tag{70}$$

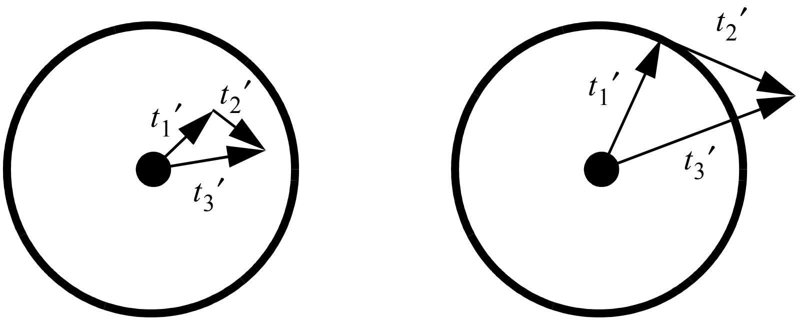
And this carries over, identically, to subtraction so that the resultant magnitude is the sum, not the difference, of the magnitudes. Consequently, absolute arithmetic is implemented on the axle at angle nullity and signed arithmetic is implemented on the cylinder, or *swept rim,* comprising all of the rims – with the wheels at zero and nullity having identified parts so that they have no positive or negative sign, as usual in real and transreal arithmetic.

$$(r_1, c_1, s_1) - (r_2, \Phi, \Phi) = (r_1 + r_2, \Phi, \Phi) \tag{71}$$

It is convenient to give the proof in polar-transcomplex form:

$$(r_1, \theta_1) - (r_2, \Phi) = (r_1, \theta_1) + (r_2, \Phi + \pi) = (r_1, \theta_1) + (r_2, \Phi) = (r_1 + r_2, \Phi) \tag{72}$$

All other cases involving one or two arguments at angle nullity are obtained similarly. Thus, the addition or subtraction of any numbers, at least one of which is at angle nullity, involves the projection of any arguments on the rim of a wheel to the hub of the wheel, followed by a translation up the axle at angle nullity to a transreal distance of $r_1 + r_2$. This geometrical operation is in addition to Newton's parallelogram rule and provides the one case of transcomplex addition that Newton does not: the sum of a number at angle nullity with a number that is not at angle nullity.



(a) (b)

Figure 9: Vector sums in the plane of a wheel.

Newton, [12] pp 111-112 & 417-418, defines what we now call the *vector sum* by a geometrical process of laying off the sum at every point along the resultant until the end point is arrived at. Whereas, the modern parallelogram rule sums from beginning to end of a vector without considering intermediate points. The modern method fails for all non-finite vectors, but Newton's method succeeds for every combination of finite and non-finite vectors, except the sum of a vector at angle nullity with a vector that is not at angle nullity. We supply this missing case above, but it is difficult to give an accurate numerical implementation of integration so, instead, we use a different geometrical construction which projects all vectors to finite length, performs modern vector addition on the transformed vectors, and then projects the resultant to its true length. This true length may be non-finite. We now describe this construction by working through the top-level "otherwise" branch of Equation (66), relating this to Figure 4, Figure 8 and Figure 9.

Having already dealt with a vector or vectors at angle nullity, we now deal with vectors at finite angles. The transcomplex number $t_1 = (r_1, c_1, s_1)$ lies on the swept wheel, Figure 8, at height $z_1 = r_1$. It lies at angle θ_1 on the rim of this wheel. Compare with the polar-transcomplex points $(0, 0)$, (r, θ), (∞, θ) in Figure 4. Similarly, $t_2 = (r_2, c_2, s_2)$ lies on the wheel at height $z_2 = r_2$. It lies at angle θ_2 on the rim of this wheel. The point t_1 is projected orthogonally along the z-axis as t_1' onto the plane of the wheel at height $z = r_1 + r_2$. If $r_1 = r_1 + r_2$ then the vector t_1' is of unit length. Otherwise t_1' is of length $r_1 \div (r_1 + r_2)$. Similarly, the point t_2 is projected orthogonally as t_2' onto the plane of the wheel at height $z = r_1 + r_2$. Two such configurations are shown in Figure 9 (a), (b). With this construction all of the projected vectors are of finite length, all information about the original lengths is encoded, and the original angles are preserved. The vectors t_1', t_2' are then summed using the modern parallelogram rule giving a resultant $t_3' = (r_3', c_3, s_3)$, where c_3, s_3 are clamped to the angle zero if $r_3' = 0$. The vector t_3' is then scaled to its final magnitude by $r_3 = r_3' \times (r_1 + r_2)$ and is written into the required wheel. If $r_3 \in \{0, \Phi\}$ then the angle collapses, as given by identification within the wheels at heights zero and nullity.

One subtlety of this geometrical construction is that it correctly computes transreal $\infty - \infty = \Phi$ and, more generally, it computes transcomplex $(\infty, \theta) - (\infty, \theta) = (\Phi, \Phi)$. The reason is that $t_1 - t_2 = (\infty, \theta) - (\infty, \theta)$ projects orthogonally onto the wheel at height $z = r_1 + r_2 = \infty$ with $t_3' = (1, \theta) - (1, \theta) = (0, \theta)$ whence $r_3 = r_3' \times (r_1 + r_2) = 0 \times \infty = \Phi$, as required.

When floating-point arithmetic is used, the above calculations generally have some rounding error so that the lengths of computed vectors and their angles are approximate, but bounded. When vectors round off to infinite length, the error is bounded by infinity – which is an exceptionally loose bound. But if the length or angle round off to nullity then the error is neither small nor large, but it is unbounded. An error of nullity represents a maximal loss of information. Such is the case, for example, when vectors of infinite length and nearly opposite angles are added so that their sum rounds off to the nullity vector; or where a vector of exactly infinite length is multiplied by a vector of inexact (underflowed) length zero. However, it is sometimes possible to recover some of the information about length or magnitude by tracing the computational paths that produced the inexact finite or non-finite numbers. And wherever this strategy succeeds, it would be possible to re-work the algorithm so that it responds to inexact flags, in its floating-point values, at appropriate points in a path or confluence of paths. Therefore, it is important that all floating-point numbers, whether finite or non-finite, carry an inexact flag through a computation until the inexactness can be used or reported. This is an improvement on IEEE floating-point arithmetic in that all floating-point objects carry diagnostic information, rather than having a hardware implementation provide this facility only for NaNs. Having an inexact flag universally available might encourage programmers to producer maximally accurate programs.

Note that it is possible to compute exactly opposite angles in floating-point arithmetic using the formulae $\theta_1 = \pi \div 2 + e$ and $\theta_2 = -\pi \div 2 + e$, where π is the floating-point representation of pi and e is an arbitrary angle. Here the opposition of angles is carried explicitly in the sign bit. The magnitudes, and therefore the significands, of $\pi \div 2$ and $-\pi \div 2$ are equal and e is identically equal. Hence the opposition is exact for all real values of e. All non-finite values of e result in an angle of nullity so opposition is exact in this case, too. Hence, opposition is exact for all transreal e. (However, implemented trigonometrical functions do not necessarily preserve this exact opposition. See the on-line appendices.)

The reader may find that having a geometrical interpretation of the arithmetical operators aids in their understanding and application. One should not forget, however, that topological considerations also apply, as in our earlier discussion of physical singularities and limits in analysis. This is no surprise. Real arithmetic also requires such consideration for its proper application, if it applies at all.

8.6 Relationship of Transcomplex Arithmetic to Other Arithmetics

Having introduced algebraic and geometrical constructions of the basic arithmetical operators of addition, subtraction, multiplication, and division, we now show how transcomplex arithmetic relates to other arithmetics.

Complex numbers, z, are usually defined in terms of *Cartesian-complex* numbers, $z = (x, y) = x + iy$, with $i = \sqrt{-1}$ being the unit complex-vector. This construction has an implicit unit real-vector, r, so that Cartesian-complex numbers are given more explicitly by $z = (x, y) = rx + iy$ with x, y being dimensionless, that is scalar, numbers. However, in what follows, we use the more popular, and more compact, form $z = (x, y) = x + iy$, recalling, where necessary, that x has dimension real and y is dimensionless. In what follows, the symbol r denotes a radius, as it has done in most of this paper, and r, c, s are, respectively, the radius, cosine and sine of a transcomplex number (r, c, s).

With the usual notation, [7] [8], *Cartesian-complex* numbers, (x, y), are given by (rc, rs) with x on the real axis and y on the imaginary axis; *polar-complex* numbers, (r, θ), in the principal range, $-\pi < \theta \le \pi$, are given by $(r, \text{arctan2}(c, s))$; *Eulerian-complex* numbers, $n = re^{i\theta}$, in the above principal range, are given by $n = (rc, irs)$, where i is the complex unit, $(1, 0, 1)$, in three-tuple form; and *Riemannian-complex* numbers, being all of the Cartesian-complex, polar-complex, and Eulerian-complex numbers, with *complex infinity,* that arise from projection of the Riemann sphere, [8] [9] [10], are described by Cartesian-complex, polar-complex, or Eulerian-complex numbers, together with the point at infinite radius and nullity angle, (∞, Φ, Φ), which replaces the ordinary complex infinity.

We will prove that transcomplex arithmetic implements the Cartesian-complex operations of addition, subtraction, multiplication, and division.

Polar-complex arithmetic defines multiplication as the composition of a dilatation and rotation: $(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 \times r_2, \theta_1 + \theta_2)$. Division is defined similarly. Here, $0 \le r_i \le \infty$ and θ_i may be taken to range over all real numbers so that polar-complex numbers and their products and quotients lie somewhere on a Riemann surface. Alternatively, θ_i may be taken in a principal range, for example $-\pi < \theta_i \le \pi$, so that polar-complex numbers and their products and quotients fill out the whole of the complex plane. A particular difficulty with polar-complex form is that when the radius is zero, the angle is undefined, in other words, the angle is a real variable. Transcomplex arithmetic avoids this difficulty, at zero radius, by identifying all finite angles with the angle zero. Polar-complex form does not define operations of addition and subtraction. Instead polar-complex numbers are cast to Cartesian-complex form, are added or subtracted, and the result is cast back to polar-complex form. This round-trip of casts generally involves the computation of both trigonometrical and arc-trigonometrical functions, making the operation slow and inaccurate on a digital computer. It is already known that polar-complex arithmetic, in a principal range, is a proper subset of Cartesian-complex arithmetic so nothing remains for us to prove here.

Eulerian-complex arithmetic defines multiplication as, $n_1 \times n_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = (r_1 \cos\theta + r_1 i \sin\theta) \times (r_2 \cos\theta + r_2 i \sin\theta)$. Division is defined similarly. Addition is defined as $n_1 + n_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2} = (r_1 \cos\theta + r_1 i \sin\theta) + (r_2 \cos\theta + r_2 i \sin\theta)$, and subtraction is defined similarly. Here $0 \le r_i \le \infty$ so that Eulerian-complex numbers and their products, quotients, sums, and differences fill out the whole of the complex plane. In every case the arithmetical operations are Cartesian-complex, as shown by the right most part of the above equations, so nothing remains for us to prove here.

The Riemannian-complex numbers may be presented as abstract projections which make no particular commitment to the form of complex-co-ordinates. However, projection of the Riemann sphere does fill out the whole of the complex plane so that arithmetic in this plane is obliged to be Cartesian-complex or isomorphic to it. We deal with this case in our proof that transcomplex arithmetic implements Cartesian-complex arithmetic. However, we must also deal with arithmetic on complex infinity. Greenleaf gives the required operations in Equation (47), but as transreal arithmetic is defined to be totally commutative [1] we need only prove the following, where complex infinity, ∞, is transcomplex (∞, Φ, Φ). Note that transcomplex arithmetic is a proper superset of Riemannian-complex arithmetic because (Φ, Φ, Φ) is transcomplex, but is not Riemannian-complex.

$$\begin{aligned} \infty \times z &= \infty && \textit{for } z \neq 0 \textit{ in } \boldsymbol{C} \\ \infty + z &= \infty && \textit{for all } z \textit{ in } \boldsymbol{C} \\ \infty - z &= \infty && \textit{for all } z \textit{ in } \boldsymbol{C} \\ z/0 &= \infty && \textit{for } z \neq 0 \textit{ in } \boldsymbol{C} \\ z/\infty &= 0 && \textit{for all } z \textit{ in } \boldsymbol{C} \\ \infty \times \infty &= \infty \end{aligned} \tag{73}$$

We also wish to prove that transreal arithmetic is a proper subset of transcomplex arithmetic. It is trivial to prove the proper part: $(\infty, \sqrt{2}, \sqrt{2})$ is transcomplex but not transreal. The finite part is given by the proof that transcomplex arithmetic implements Cartesian-complex arithmetic, because real arithmetic is a proper subset of complex arithmetic. Some of the infinite part is given by proving Equations (73) where the symbol ∞ denotes transreal infinity in transcomplex form: $(\infty, 1, 0)$. The remaining non-finite part is given by proving the Equations (74), with the same interpretation of the symbol ∞, and where the symbol Φ denotes transcomplex nullity (Φ, Φ, Φ). Compare with Equations (48).

When all of this is proved we will have shown that transcomplex arithmetic is a universal complex arithmetic in as much as it contains the above arithmetics in the principal range.

$$
\begin{aligned}
\infty + \infty &= \infty \\
\infty - \infty &= \Phi \\
\infty \div \infty &= \Phi \\
0 \div 0 &= \Phi \\
\infty \times 0 &= \Phi \\
\Phi + z &= \Phi \qquad \text{for all } z \text{ in } R^T \\
\Phi - z &= \Phi \qquad \text{for all } z \text{ in } R^T \\
\Phi \times z &= \Phi \qquad \text{for all } z \text{ in } R^T \\
\Phi \div z &= \Phi \qquad \text{for all } z \text{ in } R^T
\end{aligned} \tag{74}
$$

8.7 Proof

In this subsection, we give the proofs identified in the previous subsection. Here p is an arbitrary, fixed, positive, real number: $0 < p < \infty$. Consequently, the real numbers equate to the transcomplex numbers, (r, c, s), as:

$$
\begin{aligned}
0 &= (0, 1, 0) \\
p &= (p, 1, 0) \\
-p &= (p, -1, 0)
\end{aligned} \tag{75}
$$

The complex unit, $i = \sqrt{-1}$, equates to a transcomplex number (r, c, s) as:

$$
i = (1, 0, 1) \tag{76}
$$

The Cartesian-complex numbers, $(x, y) = x + iy$, relate to the transcomplex numbers, (r, c, s), as:

$$
(x, y) = \begin{cases} (\sqrt{x^2 + y^2}, x \div \sqrt{x^2 + y^2}, y \div \sqrt{x^2 + y^2}) \;:\; 0 < \sqrt{x^2 + y^2} < \infty \\ (\sqrt{x^2 + y^2}, 1, 0) \;:\; \sqrt{x^2 + y^2} = 0 \end{cases} \tag{77}
$$

When presenting proofs, we use the ordinary conventions of arithmetic. Multiplication of expressions, other than digit strings, may be written implicitly, $x \times y = xy$. Division may be written using a obelus, vinculum, or else a solidus, each with its usual bracketing: $(x) \div (y) = \frac{x}{y} = (x)/(y)$.

8.7.1 Cartesian-Complex Multiplication

It is to be proved that transcomplex multiplication implements Cartesian-complex multiplication.

Cartesian-complex multiplication has:

$$
(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \tag{78}
$$

Whence:

$$\begin{aligned} r_C &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\ &= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1x_2y_1y_2 + x_2^2y_1^2} \\ &= \sqrt{x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2} \end{aligned} \tag{79}$$

Now, for all products that do not involve two zero factors, we have $r_C \neq 0$ and:

$$c_C = \frac{x_1x_2 - y_1y_2}{r_C} \tag{80}$$

$$s_C = \frac{x_1y_2 + x_2y_1}{r_C} \tag{81}$$

Lemma: let $r_1 = \sqrt{x_1^2 + y_1^2}$ when $r_2 = \sqrt{x_2^2 + y_2^2}$ then:

$$\begin{aligned} r_1r_2 &= \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \\ &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ &= \sqrt{x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2} \\ &= r_C \end{aligned} \tag{82}$$

This completes the lemma. Now, returning to the proof, the corresponding transcomplex multiplication of non-zero factors is:

$$\begin{aligned} (x_1 + iy_1)(x_2 + iy_2) &= \left(\sqrt{x_1^2 + y_1^2}, \frac{x_1}{\sqrt{x_1^2 + y_1^2}}, \frac{y_1}{\sqrt{x_1^2 + y_1^2}}\right)\left(\sqrt{x_2^2 + y_2^2}, \frac{x_2}{\sqrt{x_2^2 + y_2^2}}, \frac{y_2}{\sqrt{x_2^2 + y_2^2}}\right) \\ &= \left(r_1, \frac{x_1}{r_1}, \frac{y_1}{r_1}\right)\left(r_2, \frac{x_2}{r_2}, \frac{y_2}{r_2}\right) \\ &= \left(r_1r_2, \frac{x_1}{r_1}\frac{x_2}{r_2} - \frac{y_1}{r_1}\frac{y_2}{r_2}, \frac{y_1}{r_1}\frac{x_2}{r_2} + \frac{x_1}{r_1}\frac{y_2}{r_2}\right) \\ &= \left(r_1r_2, \frac{x_1x_2 - y_1y_2}{r_1r_2}, \frac{x_1y_2 + x_2y_1}{r_1r_2}\right) \\ &= \left(r_C, \frac{x_1x_2 - y_1y_2}{r_C}, \frac{x_1y_2 + x_2y_1}{r_C}\right) \\ &= (r_C, c_C, s_C) \end{aligned} \tag{83}$$

This completes the proof that transcomplex multiplication implements Cartesian-complex multiplication, not involving a zero factor. All multiplications involving one or two zero factors are obtained similarly. Q.E.D.

8.7.2 Cartesian-Complex Division

It is to be proved that transcomplex division implements Cartesian-complex division.

For all finite, non-zero denominators, $x_2 + iy_2 \neq 0$, Cartesian-complex division has:

$$\begin{aligned}
\frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\
&= \frac{x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2}{x_2^2 - ix_2y_2 + ix_2y_2 + y_2^2} \\
&= \frac{(x_1x_2 + y_1y_2) + i(-x_1y_2 + x_2y_1)}{x_2^2 + y_2^2}
\end{aligned} \tag{84}$$

Whence:

$$\begin{aligned}
r_C &= \frac{\sqrt{(x_1x_2 + y_1y_2)^2 + (-x_1y_2 + x_2y_1)^2}}{x_2^2 + y_2^2} \\
&= \frac{\sqrt{x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2}}{x_2^2 + y_2^2} \\
&= \frac{\sqrt{x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2}}{x_2^2 + y_2^2} \\
&= \frac{r_1r_2}{r_2^2} \\
&= \frac{r_1}{r_2}
\end{aligned} \tag{85}$$

Now, for all non-zero numerators, we have $r_1 \neq 0$ and:

$$\begin{aligned}
c_C &= \frac{r_2}{r_1} \times \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} \\
&= \frac{r_2}{r_1} \times \frac{x_1x_2 + y_1y_2}{r_2^2} \\
&= \frac{x_1x_2 + y_1y_2}{r_1r_2}
\end{aligned} \tag{86}$$

$$\begin{aligned}
s_C &= \frac{r_2}{r_1} \times \frac{-x_1y_2 + x_2y_1}{x_2^2 + y_2^2} \\
&= \frac{r_2}{r_1} \times \frac{-x_1y_2 + x_2y_1}{r_2^2} \\
&= \frac{-x_1y_2 + x_2y_1}{r_1r_2}
\end{aligned} \tag{87}$$

The corresponding transcomplex division with a non-zero numerator, $x_1 + iy_1 \neq 0$, is:

$$\begin{aligned}
\frac{x_1 + iy_1}{x_2 + iy_2} &= \left(r_1, \frac{x_1}{r_1}, \frac{y_1}{r_1}\right) \div \left(r_2, \frac{x_2}{r_2}, \frac{y_2}{r_2}\right) \\
&= \left(r_1 \div r_2, \frac{x_1 x_2}{r_1 r_2} + \frac{y_1 y_2}{r_1 r_2}, \frac{y_1 x_2}{r_1 r_2} - \frac{x_1 y_2}{r_1 r_2}\right) \\
&= \left(r_1 \div r_2, \frac{x_1 x_2 + y_1 y_2}{r_1 r_2}, \frac{-x_1 y_2 + x_2 y_1}{r_1 r_2}\right) \\
&= (r_C, c_C, s_C)
\end{aligned} \tag{88}$$

This completes the proof that transcomplex division implements Cartesian-complex division, not involving a zero numerator. Division involving a zero numerator is obtained similarly. Q.E.D.

8.7.3 Cartesian-Complex Addition and Subtraction

It is to be proved that transcomplex addition and subtraction implement Cartesian-complex addition and subtraction.

Transcomplex subtraction is defined in terms of addition, see Equation (67), so that every subtraction becomes an addition:

$$\begin{aligned}
(r_1, c_1, s_1) - (r_2, c_2, s_2) &= (r_1, c_1, s_1) + (1, -1, 0) \times (r_2, c_2, s_2) \\
&= (r_1, c_1, s_1) + (1r_2, -1c_2 - 0s_2, 0c_2 - 1s_2) \\
&= (r_1, c_1, s_1) + (r_2, -c_2, -s_2)
\end{aligned} \tag{89}$$

Consequently, it is necessary to prove only that transcomplex addition implements Cartesian-complex addition. We note, in passing, that the relationship just derived, $(r_1, c_1, s_1) - (r_2, c_2, s_2) = (r_1, c_1, s_1) + (r_2, -c_2, -s_2)$, is more succinct than Equation (67) so it might be preferred as a definition.

Cartesian-complex addition has:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \tag{90}$$

Whence:

$$r_C = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \tag{91}$$

Now, for non-zero sums we have $r_C \neq 0$ and:

$$c_C = \frac{x_1 + x_2}{r_C} \tag{92}$$

$$s_C = \frac{y_1 + y_2}{r_C} \tag{93}$$

The corresponding transcomplex addition, with a non-zero sum, has the single side condition $(x_1 + iy_1) \neq -(x_2 + iy_2)$, which implies $r_1 + r_2 \neq 0$. Now:

$$(x_1 + iy_1) + (x_2 + iy_2) = \left(r_1, \frac{x_1}{r_1}, \frac{y_1}{r_1}\right) + \left(r_2, \frac{x_2}{r_2}, \frac{y_2}{r_2}\right) \tag{94}$$

Whence, observing the above side condition, Equation (66) gives:

$$r_1' = \frac{r_1}{r_1 + r_2} \tag{95}$$

$$r_2' = \frac{r_2}{r_1 + r_2} \tag{96}$$

$$\begin{aligned} x &= r_1'c_1 + r_2'c_2 \\ &= \frac{r_1}{r_1 + r_2} \times \frac{x_1}{r_1} + \frac{r_2}{r_1 + r_2} \times \frac{x_2}{r_2} \\ &= \frac{x_1 + x_2}{r_1 + r_2} \end{aligned} \tag{97}$$

Similarly:

$$y = \frac{y_1 + y_2}{r_1 + r_2} \tag{98}$$

Now:

$$\begin{aligned} r_3' &= \sqrt{x^2 + y^2} \\ &= \sqrt{\left(\frac{x_1 + x_2}{r_1 + r_2}\right)^2 + \left(\frac{y_1 + y_2}{r_1 + r_2}\right)^2} \\ &= \frac{1}{r_1 + r_2}\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \end{aligned} \tag{99}$$

$$\begin{aligned} r_3 &= r_3'(r_1 + r_2) \\ &= \frac{1}{r_1 + r_2}\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}(r_1 + r_2) \\ &= \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \\ &= r_C, \text{ as required.} \end{aligned} \tag{100}$$

And:

$$r_3' = \frac{r_3}{r_1 + r_2} \tag{101}$$

Now:

$$\begin{aligned} c_3 &= \frac{x}{r_3}, \\ &= \frac{x_1 + x_2}{r_1 + r_2} \times \frac{r_1 + r_2}{r_3} \\ &= \frac{x_1 + x_2}{r_3} \\ &= c_C, \text{ as required.} \end{aligned} \tag{102}$$

Similarly, $s_3 = s_C$, as required.

This completes the proof that transcomplex addition implements both Cartesian-complex addition and subtraction, not involving zero sums. Zero sums are obtained similarly. Q.E.D.

8.7.4 Arithmetic Involving Riemannian-Complex Infinity

It is to be proved that Transcomplex arithmetic implements Riemannian-complex arithmetic. Given the proofs above, it remains only to prove that arithmetic on complex infinity is supported.

Equation (73) give $\infty \times z = \infty$ for complex $z \neq 0$. Taking complex infinity, ∞, as transcomplex infinity at angle nullity, (∞, Φ, Φ), gives:

$$(\infty, \Phi, \Phi)(r_z, c_z, s_z) = (\infty r_z, \Phi c_z - \Phi s_z, \Phi c_z + \Phi s_2) = (\infty, \Phi, \Phi)\text{, as required.} \tag{103}$$

The remaining cases of Equation (73) are obtained similarly. Q.E.D.

8.7.5 Transreal Arithmetic

It is to be proved that transcomplex arithmetic implements transreal arithmetic. Given the proofs above, it remains only to prove the non-finite cases.

Equation (73) give $\infty \times z = \infty$. Taking the symbol, ∞, as transreal infinity, being transcomplex infinity on the real axis, $(\infty, 1, 0)$, and taking z as real and strictly positive, being transcomplex and non-zero on the real axis, $(z, 1, 0)$, gives:

$$(\infty, 1, 0)(z, 1, 0) = (\infty z, 1 \times 1 - 0 \times 0, 0 \times 1 + 1 \times 0) = (\infty, 1, 0)\text{, as required.} \tag{104}$$

The remaining cases of Equation (73) and (74) are obtained similarly. Q.E.D.

9. Transcomplex Exponential, Logarithm, and Raising to a Power

In this section we present preliminary totalisations of the exponential and logarithmic functions, and of the operation of raising a number to the power of a number.

There are two problems when generalising a power series from its ordinary form to transnumbers. The first is that we must distinguish the cases of taking the series asymptotically to infinity, that is over all natural numbered indices, and exactly to infinity, that is over all natural numbered indices and the index ∞. Secondly, we cannot always elide a term from a power series by zeroing it. For example, $a_1 x^1 = 0 \times \infty^1 = 0 \times \infty = \Phi \neq 0$. We handle this by introducing a Boolean elision operator, Ψ_i, such that the i'th term is contained in the series if $\Psi_i = 1$ and is elided from the series if $\Psi_i = 0$.

In general, we write a power series, S_i, taken up to the i'th term, as an infinite sum with all $\Psi_j = 0$ when $j > i$:

$$S_\infty = \Psi_0 a_0 1^0 + \Psi_1 a_1 x^1 + \Psi_2 a_2 x^2 + \ldots + \Psi_n a_n x^n + \Psi_\infty a_\infty x^\infty \tag{105}$$

Here i may be a finite number, an indefinitely large natural number, n, or infinity, ∞. In deference to existing notations, we write:

$$\sum_{i=0}^{i \to \infty} \Psi_i a_i x^i = S_n \text{ where the term } x^0 \text{ is substituted by } 1^0 \tag{106}$$

$$\sum_{i=0}^{i = \infty} \Psi_i a_i x^i = S_\infty \text{ where the term } x^0 \text{ is substituted by } 1^0 \tag{107}$$

With this arrangement, all transreal power series taken exactly to infinity have a fixed sum and all transcomplex power series taken exactly to infinity have a fixed sum of magnitudes, but may currently have an undefined sum of phases.

We present this general notation as an aid for the reader who wishes to explore transpower series. In the special case of the transcomplex exponential we take all $\Psi_i = 1$, except $\Psi_\infty = 0$. We also provide boundary cases to totalise the exponential.

$$\exp(r, c, s) = \begin{cases} (r, c, s) & : \; r = \Phi \vee (r = \infty \wedge \neg(c < 0 \vee s = -1)) \\ 0 & : \; r = \infty \wedge (c < 0 \vee s = -1) \\ (\exp r, c, s) & : \; c, s = \Phi \\ \displaystyle\sum_{n=0}^{n \to \infty} \frac{x^n}{n!} & : \; \text{otherwise} \end{cases} \tag{108}$$

As usual, the logarithm is defined to be the mapping from the image of the exponential function to its pre-image. For the transreal exponential, $\exp(r, 1, 0)$ and $\exp(r, -1, 0)$, this mapping is an inverse, see [6], but it is not an inverse in the transcomplex case. Nonetheless, we can exploit the assumed identity $(xy)^z = x^z y^z$ to obtain a function which raises any transcomplex number, $n = xy$, to the power of any transcomplex number z. Thus, the function $\text{pow}(x, z) = x^z$ is defined as:

$$\operatorname{pow}((r_x, c_x, s_x), z) = \begin{cases} \exp(z\log(k, c_x, s_x)) \times \exp(z\log(r_x, 1, 0)) & : r_x = \infty \\ \exp(z\log(r_x, c_x, s_x)) & : \text{otherwise} \end{cases} \tag{109}$$

Here k is any general, non-negative constant. In other words, k is finite, positive, and not equal to unity. See [6] for some special properties of unity. It is convenient to choose k equal to the radix of the floating-point system in which transcomplex arithmetic is implemented. The on-line appendices are implemented in binary floating-point arithmetic and take $k = 2$. See *trans_exp, trans_log, trans_power* in the on-line appendices.

Note that the treatment of raising to a power in [6] is now seen to be defective.

10. Discussion

We have proved, elsewhere [1], that division by zero, in transreal arithmetic, is consistent if real arithmetic is, and have noted, in the present Introduction, various other methods for dealing with division by zero. We maintain that transreal arithmetic is the pre-eminent development of real arithmetic, because all of its algorithms are universally accepted algorithms of real arithmetic. Consequently, transreal arithmetic is identical to real arithmetic in all finite computations; but where real arithmetic fails to apply, to any case involving division by zero, transreal arithmetic continues to hold. We can see no reason why anyone would prefer a partial arithmetic over a total arithmetic.

Of course, it is not enough that transreal arithmetic is consistent, it must also be useful if it is to be accepted as the natural successor to real arithmetic. Indeed, the whole of this paper can be read as an argument for the usefulness of transreal arithmetic in mathematics, computation and physics.

The first argument for usefulness is that transreal arithmetic is simple. It is so simple that it can be taught to twelve-year-old children. And transreal arithmetic is useful to children. Those who develop in mathematical and physical understanding to the point where they can apply Newton's laws of motion, find that they can solve problems involving physical singularities that currently defeat professional physicists. If transreal arithmetic is eventually accepted by society then it will be necessary to teach it to children to prepare them to take their place in the world. In the mean time, transreal arithmetic offers an interesting diversion that might be used in the classroom to motivate the study of accepted interpretations of mathematics and physics. But transreal arithmetic is a serious business, it stands as a challenge to teachers: why continue to teach children a mathematics that is guaranteed to fail in some cases, when a total arithmetic could be taught?

A second argument for the usefulness of transreal arithmetic is that it provides a natural development of ancient forms of mathematics. All of Newton's Arithmetica Naturalis ([15] vol. 5, especially pp. 52-109) can be read as applying to transreal numbers. The methods of proportions, in Book 5 of Euclid's Elements ([31] vol 2), dating back to the 3rd Century B.C., can be read as applying to transreal numbers; as can the methods of proportions, in Book 7 of Euclid's Elements ([31] vol 2), that date back to Pythagoras in the 6th Century B.C., and earlier to the Babylonians. Thus, transreal arithmetic can be used as a vehicle to introduce the history of mathematics to learners; can stand as a philosophical and historical example of monotonic theory development in science; and can reassure users that transreal arithmetic is compatible with, and an improvement on, all of the real arithmetic that went before. This might encourage others to seek further improvements in mathematics.

Transreal arithmetic is also psychologically appealing because it provides a Gestalt of Good Form. The history of mathematics can be read as the, somewhat haphazard, development of total systems of mathematics. Thus, natural arithmetic is totalised by integer arithmetic, which is totalised by rational arithmetic, which is totalised by real arithmetic, which is totalised by transreal arithmetic, which is totalised by transcomplex arithmetic, which has yet to be totalised by transquaternion and transoctonion arithmetics. But the appeal is not purely psychological, transreal arithmetic has advantages for computation, and might also have advantages for mathematical physics.

We have described, here and elsewhere [5], how transreal arithmetic makes computer arithmetic more efficient, and how it makes it easier and safer to write computer programs. We see no useful purpose in wasting 9 007 199 254 740 990 states in every double precision, IEEE floating-point core that has been manufactured to date, nor of encouraging costly and dangerous failures in the development of software by specifying a complicated numerical and non-numerical ordering when a much simpler transreal ordering is available. We look to the practical advantages of transreal computation to draw society into its use.

Some contemporary physicists assert that it is impossible to divide by zero. For them, it is mathematically impossible that the universe adopts any singular configuration so they introduce one or more *cosmic censors* which cut off every possible, physical, occurrence of division by zero. For us, division by zero is an elementary property of transreal arithmetic so we have no *a priori* reason to suppose that cosmic sensors exist. If such censors do exist, then that is a property of the physical universe which remains to be demonstrated empirically. In the mean time, we have shown how school children might carry out theoretical calculations on gravitational and electrostatic singularities using neo-Newtonian physics. We argue that as transreal arithmetic is total, it will be able to compute solutions at any singularities whatever, though, in many cases, the mathematical procedures needed to carry out the computations remain to be developed.

We also argue that transreal arithmetic is useful because it leads to the development of transcomplex arithmetic, which is useful, in turn, because it is a universal complex-arithmetic. In particular, it performs all of the roles of Cartesian-complex and Eulerian-complex arithmetics which are very popular in engineering and physics. Applications depending on the Riemann sphere and fractional linear transformations are also improved.

However, we readily acknowledge that our results relating to the transcomplex exponential and transcomplex logarithm are preliminary. Experience might cause us to revise them. To this extent, then, the operation of raising a transcomplex number to a transcomplex power is subject to revision. And, as our motive is to create a useful mathematics of division by zero, we will revise any part of transmathematics that is contradicted by experience. Thus, the whole of transmathematics is subject to revision. In particular, we might revise our decision to identify the circle at nullity with the point at nullity, and we might revise our arrangement of the addition of a vector at a finite angle with a vector at a non-finite angle.

Returning to our cautionary tale, some physicists draw grand conclusions from the supposed impossibility of dividing by zero. One such, goes like this: Einstein's theory of relativity allows gravitational singularities, these involve division by zero, but division by zero is mathematically impossible, therefore Einstein's theory of relativity is not a complete description of physical gravity. The fault in this line of reasoning is that division by zero is possible. Einstein's theories might, or might not, be an adequate description of physical gravity; but that question is to be settled by experiment, not by an armchair appeal to fanciful limitations of mathematics. We see no reason why contemporary physicists should prefer a mathematics that fails, in some cases, over a total mathematics. We invite everyone to consider Paul Dirac's views on the advancement of mathematical physics ([34], p. 60).

> *The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers of the last [19^{th}] century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modern physical developments have required a mathematics that continually shifts its foundations and gets more abstract. Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.*
>
> *– Paul Dirac [34], p. 60.*

We have modified the axioms of arithmetic so as to admit division by zero, making transreal arithmetic total [1], and more abstract than real arithmetic, in that it adopts the number nullity. We cannot see any advantage in requiring, as our contemporaries do, that mathematics, in its very foundations of arithmetic, is partial, causing it to fail to apply in some cases. We maintain that transreal arithmetic is natural and useful. Accordingly, we invite the reader to join us in making the paradigm shift to transarithmetic that allows division by zero everywhere in mathematics, computation and physics.

Now we come to the nub of the argument. There can be no compromise with readers who continue to assert that division by zero is impossible. For them, we throw down a definitive challenge. We assert that:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} \tag{110}$$

We challenge readers to say what useful purpose is served by requiring, further, that $b, c, d \neq 0$. We can see no useful purpose in requiring that the existing algorithms of real and complex arithmetic fail in the cases where any of $b, c, d = 0$. On the contrary, we maintain that it is useful to allow the whole of existing mathematics to succeed in these cases, as, we maintain, the examples in this paper, and our previous papers, have begun to show.

11. Future Work

As stated in the Introduction, there are many possible ways to develop transreal arithmetic into transcomplex arithmetic. It would be useful to know if the current development is the best available. An efficient way to examine transcomplex arithmetic would be to develop it into a transquaternion and then a transoctonion arithmetic. It might be that there is some internal constraint within this hierarchy which militates in favour or against the current development. Once all of these arithmetics have been fixed, it would be efficient to axiomatise them all and give a single machine proof of their consistency. It would then be helpful to translate the machine proof into human readable form so that it is accessible to a wide range of mathematicians.

A more open-ended approach would be to develop various areas of mathematics, looking for properties that militate for or against the current development of transcomplex arithmetic. It would be possible to direct such developments toward test cases in mathematical physics. For example, developing the vector transalgebra and transcalculus necessary to totalise Maxwell's equations ought to provide many cases for testing singular behaviour in both classical and quantum electrodynamics. Simpler tests could also be made, for example, by examining Newtonian collisions, oscillations, or once-off changes of direction at the exact moment a change of motion occurs. It might also be productive to examine the cases were cosmic censors are currently employed. Most generally, any aspect of mathematics or mathematical physics might be examined to see how it bears on transcomplex arithmetic.

A more practical approach is to develop computer systems that exploit transreal or transcomplex arithmetic, as we have done, and to compare these with ordinary computer systems. We have already presented evidence, here and in [5], that transreal arithmetic is more efficient than ordinary systems of computer arithmetic. As another example, the use of transreal and transcomplex arithmetic in theorem provers and computer algebra systems ought to enforce the proper handling of guarding clauses [24] [35].

No doubt the reader can think of many specific ways of putting transcomplex arithmetic to the test.

12. Conclusion

It is already known that transreal arithmetic is consistent if real arithmetic is. This paper argues that transreal arithmetic is useful in mathematics, physics and computation. Firstly, transreal arithmetic is so simple that it can be learned by the general reader, including secondary-school children. Secondly, transreal arithmetic supports transfloating-point arithmetic which is more efficient and safer than IEEE floating-point arithmetic. We leave it as an open question as to whether it is better for transfloating-point arithmetic to double the range of real numbers represented by the IEEE floating-point bits or whether it is better to keep the range almost the same, while halving the smallest representable, non-zero, number, thereby improving precision. We propose that transfloating-point arithmetics should reserve one bit in their numbers to operate as an *inexact* flag. Thirdly, we observe that the totality of transreal and transcomplex arithmetic might make it easier to evaluate guarding clauses in theorem provers and computer algebra systems. Fourthly, transreal arithmetic extends Euclid's methods of proportions, and extends earlier methods. Fifthly, transcomplex arithmetic unifies computations in all ordinary complex number systems and totalises them so that arithmetic may be performed on any numbers, whether finite, including zero, or non-finite. Sixthly, transreal arithmetic is a proper subset of transcomplex arithmetic, just as real arithmetic is a proper subset of complex arithmetic. Seventhly, transcomplex arithmetic provides a simpler interpretation of the point at infinity as a projection of the 'north' pole of the Riemann sphere, and develops the Riemann sphere into a total geometry over all finite and non-finite transcomplex numbers. Eighthly, transcomplex numbers reduce the need for cuts in complex domains. Ninethly, transreal analysis is a proper subset of transcomplex analysis, despite the fact that real analysis is not a subset of complex analysis. Tenthly, in contrast to the modern form of the parallelogram rule, which sums only finite forces, Newton's parallelogram rule allows the summation of finite, infinite and nullity forces, except for the sum of a nullity and a non-nullity force. We extend Newton's method to deal with this case. Eleventhly, both transreal and transcomplex arithmetic extend all of the physics in Newton's *Philosophiae Naturalis Principia Mathematica* and enable the computation of gravitational and electrostatic forces at singularities, thereby calling into question the need for cosmic censors. We suppose that modern physics will be similarly extended. Twelvethly, the introduction of division by zero is consistent with Paul Dirac's view, "… that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation." Thus, we argue, by weight of examples, that both transreal arithmetic, and its development into transcomplex arithmetic, are useful. Thirteenthly, we give an implementation of transreal and transcomplex arithmetic, as an online appendix, so that the reader may more easily explore the developments presented here.

13. Supplementary material

The on-line appendix *transarith.p* is a text file that contains an implementation of transreal and transcomplex arithmetic in Pop11 [37] [38]. The on-line appendix *transarith* is a text file that contains documentation for this package, including instructions on how to install the package on a Microsoft Windows or Unix platform. The documentation also contains a transcript showing the computation of all of the numerical examples in this journal paper.

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