Transreal Calculus

Prof. Tiago S dos Reis
Dr James Anderson FBCS CITP CSci
Agenda

• Advantages of transreal calculus
• Transdifferential calculus
• Transintegral calculus
• Value to science and society
Advantages of Transcalculus
Transcalculus

• Built on the foundation of transreal arithmetic
• Built on the foundation of transreal limits
• Allows the solution of mathematical and physical problems at singularities
• Makes mathematical software more reliable
Transreal Number Line

\[ -\infty < \Phi < \infty \]
Transderivative

If $x_0 \in \mathbb{R}$ then $f'_{\mathbb{R}_T}(x_0) = f'(x_0)$
Transderivative

If \( x_0 \in \mathbb{R} \) then
\[
f'_{\mathbb{R}}(x_0) = f'(x_0)
\]

If \( x_0 \in \{-\infty, \infty\} \) then
\[
f'_{\mathbb{R}}(x_0) = \lim_{x \to x_0} f''(x)
\]
Transderivative

If $x_0 \in \mathbb{R}$ then

$$f^{'}_{\mathbb{R}_T}(x_0) = f^{'}(x_0)$$

If $x_0 \in \{-\infty, \infty\}$ then

$$f^{'}_{\mathbb{R}_T}(x_0) = \lim_{x \to x_0} f^{'}(x)$$

Otherwise

$$f^{'}_{\mathbb{R}_T}(\Phi) = \Phi$$
Transderivative

Example

\[ \frac{d}{dx} e^x = e^x \quad \text{for all} \quad x \in \mathbb{R}^T \]
Define

\[ \lim_{{x \to x_0 \atop y \to x_0}} f(x, y) = L \]

if and only if, given an arbitrary neighbourhood, \( V \) of \( L \), there is a neighbourhood, \( U \) of \( x_0 \), such that \( f(x, y) \in V \) whenever \( x \neq y \) and \( x, y \in U \setminus \{x_0\} \)
Transderivative

\[ f : \mathbb{R}^T \rightarrow \mathbb{R}^T \] is differentiable at \( \infty \) if and only if there exists

\[
\lim_{x \to \infty, y \to \infty} \frac{f(x) - f(y)}{x - y}
\]
Transderivative

\[ f : \mathbb{R}^T \rightarrow \mathbb{R}^T \] is differentiable at \( \infty \) if and only if there exists

\[ \lim_{\substack{x \to \infty \\atop y \to \infty}} \frac{f(x) - f(y)}{x - y} \]

Whence

\[ f'_T(\infty) = \lim_{\substack{x \to \infty \\atop y \to \infty}} \frac{f(x) - f(y)}{x - y} \]
Transderivative

If $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ and there is

$$\lim_{x \to x_0} \lim_{y \to x_0} \frac{f(x) - f(y)}{x - y}$$

then $f$ is differentiable at $x_0$.
Transderivative

If \( f : \mathbb{R} \to \mathbb{R} \) is continuous at \( x_0 \in \mathbb{R} \) and there is

\[
\lim_{x \to x_0} \lim_{y \to x_0} \frac{f(x) - f(y)}{x - y}
\]

then \( f \) is differentiable at \( x_0 \)

Whence

\[
f'_{\mathbb{R}}(x_0) = \lim_{x \to x_0} \lim_{y \to x_0} \frac{f(x) - f(y)}{x - y}
\]
Transderivative

If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable then there exists

$$\lim_{x \to x_0} \lim_{y \to x_0} \frac{f(x) - f(y)}{x - y}$$
Transderivative

If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuously differentiable then there exists

\[
\lim_{x \to x_0, y \to x_0} \frac{f(x) - f(y)}{x - y}
\]

Whence

\[
f'_{\mathbb{R}^T}(x_0) = \lim_{x \to x_0, y \to x_0} \frac{f(x) - f(y)}{x - y}
\]
Transderivative

If $f : \mathbb{R}^T \rightarrow \mathbb{R}^T$ is differentiable at $x_0$ then, given an arbitrary neighbourhood, $V$ of $f'_\mathbb{R} (x_0)$, there is a neighbourhood, $U$ of $x_0$, such that $\frac{f(x) - f(y)}{x - y} \in V$ whenever $x < x_0 < y$ and $x, y \in U$
**Transderivative**

If \( f : \mathbb{R}^T \rightarrow \mathbb{R}^T \) is differentiable at \( x_0 \) then, given an arbitrary neighbourhood, \( V \) of \( f_\mathbb{R}^T(x_0) \), there is a neighbourhood, \( U \) of \( x_0 \), such that \( \frac{f(x) - f(y)}{x - y} \in V \) whenever \( x < x_0 < y \) and \( x, y \in U \).

Whence \( f_\mathbb{R}^T(x_0) = \lim_{x \to x_0, \ y \to x_0} \frac{f(x) - f(y)}{x - y} \).
Transintegral

\[(a,b) := \{x \in \mathbb{R}^T; a < x < b\}\]

\[(a,b] := (a,b) \cup \{b\}\]

\[[a,b) := \{a\} \cup (a,b)\]

\[[a,b] := \{a\} \cup (a,b) \cup \{b\}\]
We could define \([a,b] = \{x \in \mathbb{R}^T; \ a \leq x \leq b\}\)

but then we would have \([a,\Phi] = \emptyset\)

We prefer our definition which gives

\([a,\Phi] = \{a,\Phi\}\)
Transintegral

\[ |I| := \begin{cases} 
0 & \text{, if } I = \emptyset \\
 k - k & \text{, if } I = \{k\} \text{ for some } k \in \mathbb{R}^T \\
 b - a & \text{, otherwise}
\end{cases} \]
We say $\chi_A$ is the characteristic function of a set, $A$, if and only if

$$
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}
$$
A set, \( P = (x_0, \ldots, x_n) \), is said to be a partition of \([a, b]\), if and only if \( x_0, \ldots, x_n \in [a, b], x_0 = a, x_n = b \)

and

\[
\begin{cases}
    \text{if } n = 2 \text{ then } x_0 \leq x_1 \\
    \text{if } n > 2 \text{ then } x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n
\end{cases}
\]
Transintegral

\[ \varphi : [a,b] \rightarrow \mathbb{R}^T \text{ is a step function, if and only if} \]

there is a partition, \( P = (x_0, \ldots, x_n) \) of \([a,b]\), and \( c_1, \ldots, c_n \in \mathbb{R}^T \), such that

\[ \varphi = \sum_{j=1}^{n} c_j I_{I_j}, \]

where \( I_j = (x_{j-1}, x_j] \) for all \( j \in \{1, \ldots, n\} \)
The set of step functions on $[a,b]$ is $S([a,b])$
Transintegral

Let \( a, b \in \mathbb{R}^T \) and let \( \varphi = \sum_{j=1}^{n} c_j X_{I_j} \) be a step function on \([a,b]\). Then the integral in \( \mathbb{R}^T \), of \( \varphi \) on \([a,b]\), is

\[
\int_{a}^{b} \varphi(x) \, dx := \sum_{j=1}^{n} c_j |I_j|
\]
Transintegral

In transreal numbers, $\not<$ is not equivalent to $\geq$

For example $\Phi \not< 0$ but $\Phi \geq 0$ does not hold
$f : [a,b] \rightarrow \mathbb{R}^T$ is integrable in $\mathbb{R}^T$ on $[a,b]$, if and only if

\[
\inf \left\{ \int_a^b \phi(x) \, dx ; \phi \in S([a,b]) \text{ and } \phi \prec f \right\} = \\
\sup \left\{ \int_a^b \sigma(x) \, dx ; \sigma \in S([a,b]) \text{ and } f \prec \sigma \right\}
\]
Transintegral

Whence the integral of $f$ in $\mathbb{R}^T$ on $[a,b]$ is

$$\int_{a}^{b} f(x) \, dx := \inf \left\{ \int_{a}^{b} \varphi(x) \, dx ; \varphi \in S([a,b]) \text{ and } \varphi \nleq f \right\}$$
Let $a, b \in \mathbb{R}$ and let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then $f$ is Riemann integrable in $\mathbb{R}$, if and only if $f$ is integrable in $\mathbb{R}^T$. 
Transintegral

Let \( a, b \in \mathbb{R} \) and let \( f : [a, b] \to \mathbb{R} \) be a bounded function. Then \( f \) is Riemann integrable in \( \mathbb{R} \), if and only if \( f \) is integrable in \( \mathbb{R}^T \)

\[
\int_a^b f(x) \, dx = \int_{\mathbb{R}^T} f(x) \, dx
\]
Transintegral

Let $f : [-\infty, \infty] \rightarrow \mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $(-\infty, \infty)$. The improper Riemann integral $\int_{-\infty}^{\infty} |f(x)| \, dx$ exists if and only if $f$ is integrable in $\mathbb{R}^T$.
Let $f : [\infty, \infty] \rightarrow \mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $(-\infty, \infty)$. The improper Riemann integral $\lim_{a \to \infty} \int_{-\infty}^{a} |f(x)| \, dx$ exists if and only if $f$ is integrable in $\mathbb{R}^T$

$$\int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to \infty} \int_{-\infty}^{a} f(x) \, dx$$
Let \( f : [\mathbb{R}] \rightarrow \mathbb{R} \) be a function that is Riemann integrable on every closed subinterval of \((\mathbb{R})\). The improper Riemann integral \( \int_{-\infty}^{\infty} |f(x)| \, dx \) exists if and only if \( f \) is integrable in \( \mathbb{R}^T \)

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{\mathbb{R}^T} f(x) \, dx
\]

In future work we do not need absolute convergence.
Let $a, b \in \mathbb{R}$ and let $f : [a, b] \rightarrow \mathbb{R}^T$ be a function such that $f((a,b]) \subset \mathbb{R}, f(a) = \infty$ and $f$ is Riemann integrable on any subinterval in $(a,b]$. Then the Riemann integral, $\int_a^b |f| (x) \, dx$, exists, if and only if $f$ is integrable in $\mathbb{R}^T$.
Transintegral

Let \( a, b \in \mathbb{R} \) and let \( f : [a, b] \rightarrow \mathbb{R}^T \) be a function such that \( f((a, b]) \subset \mathbb{R}, f(a) = \infty \) and \( f \) is Riemann integrable on any subinterval in \((a, b]\). Then the Riemann integral, \( \int_a^b |f| \, dx \), exists, if and only if \( f \) is integrable in \( \mathbb{R}^T \)

\[
\int_a^b f(x) \, dx = \int_{\mathbb{R}^T} f(x) \, dx
\]
Transintegral

Let \( a, b \in \mathbb{R} \) and let \( f : [a, b] \rightarrow \mathbb{R}^T \) be a function such that \( f((a,b]) \subseteq \mathbb{R}, f(a) = \infty \) and \( f \) is Riemann integrable on any subinterval in \((a,b] \). Then the Riemann integral, \( \int_a^b f(x) \, dx \), exists, if and only if \( f \) is integrable in \( \mathbb{R}^T \)

\[
\int_a^b f(x) \, dx = \int_{\mathbb{R}^T}^b f(x) \, dx
\]

In future work we do not need absolute convergence
Transintegral

Example

If $a \in \mathbb{R}$ and $f(a) \in \mathbb{R}$ then

$$\int_{a}^{\infty} f(x) \, dx = 0$$
Example

If \( a \in \{-\infty, \infty, \Phi\} \) then

\[
\int_{a}^{\infty} f(x) \, dx = \Phi
\]
Transintegral

Example

\[ \int_{R^T}^{a} f(x) \, dx = \int_{R^T}^{\Phi} f(x) \, dx = \Phi \]
Conclusion

• Transreal derivatives extend real derivatives

• Transreal integrals extend real integrals

• It is known that Newton’s laws of motion extend to transarithmetic and transcalculus
Value
Reach and Reliability

- Transcalculus allows the solution of mathematical and physical problems at singularities
- Makes mathematical software more reliable
Transcalculus is a very good idea