Transreal Calculus

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Agenda

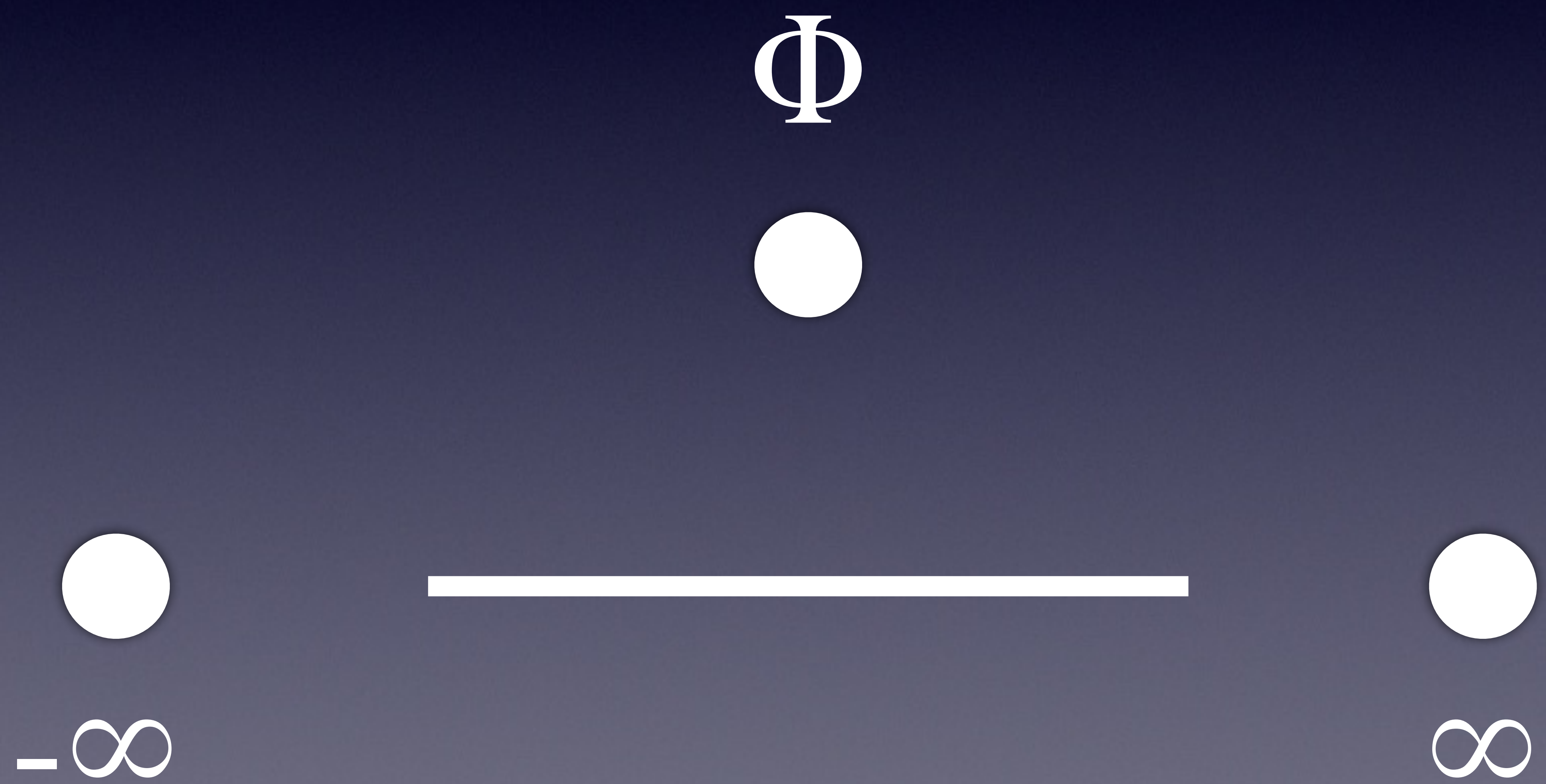
- Advantages of transreal calculus
- Transdifferential calculus
- Transintegral calculus
- Value to science and society

Advantages of Transcalculus

Transcalculus

- Built on the foundation of transreal arithmetic
- Built on the foundation of transreal limits
- Allows the solution of mathematical and physical problems at singularities
- Makes mathematical software more reliable

Transreal Number Line



Transderivative

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Transderivative

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If $x_0 \in \{-\infty, \infty\}$ then $f'_{\mathbb{R}^T}(x_0) = \lim_{x \to x_0} f'(x)$

Otherwise $f'_{\mathbb{R}^T}(\Phi) = \Phi$

Transderivative

Example

$$\frac{d}{dx}e^x = e^x \text{ for all } x \in \mathbb{R}^T$$

Transderivative

Define

$$\lim_{\substack{x \to x_0 \\ y \to x_0}} f(x,y) = L$$

if and only if, given an arbitrary neighbourhood, V of L, there is a neighbourhood, U of x_0, such that $f(x,y) \in V$ whenever $x \neq y$ and $x, y \in U \setminus \{x_0\}$

Transderivative

$f : \mathbb{R}^T \to \mathbb{R}^T$ is differentiable at ∞ if and only if there exists

$$\lim_{\substack{x \to \infty \\ y \to \infty}} \frac{f(x) - f(y)}{x - y}$$

Transderivative

$f:\mathbb{R}^T \to \mathbb{R}^T$ is differentiable at ∞ if and only if there exists

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{f(x)-f(y)}{x-y}$$

Whence

$$f'_{\mathbb{R}^T}(\infty) = \lim_{\substack{x\to\infty \\ y\to\infty}} \frac{f(x)-f(y)}{x-y}$$

Transderivative

If $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x_0\in\mathbb{R}$ and there is

$$\lim_{\substack{x\to x_0\\ y\to x_0}}\frac{f(x)-f(y)}{x-y}$$ then f is differentiable at x_0

Transderivative

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Whence

$$f'_{\mathbb{R}^T}(x_0)=\lim_{\substack{x\to x_0\\ y\to x_0}}\frac{f(x)-f(y)}{x-y}$$

Transderivative

If $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable then

there exists $\lim\limits_{\substack{x\to x_0\\ y\to x_0}} \dfrac{f(x)-f(y)}{x-y}$

Transderivative

If $f:\mathbb{R}\rightarrow\mathbb{R}$ is continuously differentiable then

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Whence

$$f'_{\mathbb{R}^T}(x_0)=\lim_{\substack{x\rightarrow x_0\\ y\rightarrow x_0}} \frac{f(x)-f(y)}{x-y}$$

Transderivative

If $f:\mathbb{R}^T \to \mathbb{R}^T$ is differentiable at x_0 then, given an arbitrary neighbourhood, V of $f'_{\mathbb{R}^T}(x_0)$, there is a neighbourhood, U of x_0, such that $\dfrac{f(x)-f(y)}{x-y} \in V$ whenever $x < x_0 < y$ and $x, y \in U$

Transderivative

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whenever $x < x_0 < y$ and $x, y \in U$

Whence $$f'_{\mathbb{R}^T}(x_0) = \lim_{\substack{x\to x_0 \\ y\to x_0}} \frac{f(x)-f(y)}{x-y}$$

Transintegral

$$(a,b) \coloneqq \{x \in \mathbb{R}^T ; a < x < b\}$$

$$(a,b] \coloneqq (a,b) \cup \{b\}$$

$$[a,b) \coloneqq \{a\} \cup (a,b)$$

$$[a,b] \coloneqq \{a\} \cup (a,b) \cup \{b\}$$

Transintegral

We could define $[a,b] = \{x \in \mathbb{R}^T ; a \le x \le b\}$

but then we would have $[a,\Phi] = \varnothing$

We prefer our definition which gives

$$[a,\Phi] = \{a,\Phi\}$$

Transintegral

$$
|I| := \begin{cases} 0 & \text{, if } I = \varnothing \\ k - k & \text{, if } I = \{k\} \text{ for some } k \in \mathbb{R}^T \\ b - a & \text{, otherwise} \end{cases}
$$

Transintegral

We say χ_A is the *characteristic function* of a set, A, if and only if

$$\chi_A(x) = \begin{cases} 1 & , \text{ if } x \in A \\ 0 & , \text{ if } x \notin A \end{cases}$$

Transintegral

A set, $P=(x_0,\ldots,x_n)$, is said to be a *partition* of $[a,b]$, if and only if $x_0,\ldots,x_n \in [a,b], x_0=a, x_n=b$

and $\begin{cases} \text{if } n=2 \text{ then } x_0 \le x_1 \\ \text{if } n>2 \text{ then } x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n \end{cases}$

Transintegral

$\varphi:[a,b]\to\mathbb{R}^T$ is a *step function,* if and only if there is a partition, $P=(x_0,\ldots,x_n)$ of $[a,b]$, and $c_1,\ldots,c_n\in\mathbb{R}^T$, such that

$$\varphi=\sum_{j=1}^{n}c_jX_{I_j},$$

where $I_j=(x_{j-1},x_j]$ for all $j\in\{1,\ldots,n\}$

Transintegral

The set of step functions on $[a,b]$ is $S([a,b])$

Transintegral

Let $a,b \in \mathbb{R}^T$ and let $\varphi = \sum_{j=1}^{n} c_j X_{I_j}$ be a step function

on $[a,b]$. Then the *integral in* $\mathbb{R}^T$, of φ on $[a,b]$, is

$$\int_{\mathbb{R}^T a}^{b} \varphi(x)\, dx := \sum_{\substack{j=1 \\ j;\, c_j \neq 0}}^{n} c_j \left| I_j \right|$$

Transintegral

In transreal numbers, $\nless$ is not equivalent to $\geq$

For example $\Phi \nless 0$ but $\Phi \geq 0$ does not hold

Transintegral

$f:[a,b]\to\mathbb{R}^T$ is *integrable in* $\mathbb{R}^T$ *on* $[a,b]$, if and only if

$$\inf\left\{\underset{\mathbb{R}^T}{\int_a^b}\varphi(x)\,dx;\ \varphi\in S([a,b])\text{ and }\varphi\not<f\right\}=$$

$$\sup\left\{\underset{\mathbb{R}^T}{\int_a^b}\sigma(x)\,dx;\ \sigma\in S([a,b])\text{ and }f\not<\sigma\right\}$$

Transintegral

Whence the *integral of* f *in* $\mathbb{R}^T$ *on* $[a,b]$ is

$$\underset{\mathbb{R}^T}{\int_a^b} f(x)\,dx \coloneqq \inf\left\{ \underset{\mathbb{R}^T}{\int_a^b} \varphi(x)\,dx;\ \varphi \in S([a,b]) \text{ and } \varphi \nless f \right\}$$

Transintegral

Let $a,b \in \mathbb{R}$ and let $f:[a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable in $\mathbb{R}$, if and only if f is integrable in $\mathbb{R}^T$

Transintegral

Let $a, b \in \mathbb{R}$ and let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable in $\mathbb{R}$, if and only if f is integrable in $\mathbb{R}^T$

$$\int_a^b f(x)\,dx = \int_{a \atop \mathbb{R}^T}^{b} f(x)\,dx$$

Transintegral

Let $f:[-\infty,\infty]\to\mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $(-\infty,\infty)$. The improper Riemann integral $\int_{-\infty}^{\infty}|f|(x)\,dx$ exists if and only if f is integrable in $\mathbb{R}^T$

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$$\int_{-\infty}^{\infty} f(x)\,dx = \underset{\mathbb{R}^T}{\int_{-\infty}^{\infty}} f(x)\,dx$$

In future work we do not need absolute convergence

Transintegral

Let $a,b \in \mathbb{R}$ and let $f:[a,b] \to \mathbb{R}^T$ be a function such that $f((a,b]) \subset \mathbb{R}, f(a) = \infty$ and f is Riemann integrable on any subinterval in $(a,b]$. Then the Riemann integral, $\int_a^b |f|(x)\, dx$, exists, if and only if f is integrable in $\mathbb{R}^T$

Transintegral

Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}^T$ be a function such that $f((a, b]) \subset \mathbb{R}, f(a) = \infty$ and f is Riemann integrable on any subinterval in $(a, b]$.

Then the Riemann integral, $\int_a^b |f|(x)\, dx$, exists, if and only if f is integrable in $\mathbb{R}^T$

$$\int_a^b f(x)\, dx = \underset{\mathbb{R}^T}{\int_a^b} f(x)\, dx$$

Transintegral

Let $a,b \in \mathbb{R}$ and let $f:[a,b] \to \mathbb{R}^T$ be a function such that $f((a,b]) \subset \mathbb{R}, f(a) = \infty$ and f is Riemann integrable on any subinterval in $(a,b]$. Then the Riemann integral, $\int_a^b |f|(x)\,dx$, exists, if and only if f is integrable in $\mathbb{R}^T$

$$\int_a^b f(x)\,dx = \int_{\mathbb{R}^T a}^{b} f(x)\,dx$$

In future work we do not need absolute convergence

Transintegral

Example

If $a \in \mathbb{R}$ and $f(a) \in \mathbb{R}$ then

$$\int_{a}^{a}{}_{\mathbb{R}^T} f(x)\, dx = 0$$

Transintegral

Example

If $a \in \{-\infty, \infty, \Phi\}$ then

$$\underset{\mathbb{R}^T}{\int_a^a} f(x)\,dx = \Phi$$

Transintegral

Example

$$\int_{a}^{\Phi}_{\mathbb{R}^T} f(x)\,dx = \int_{\Phi}^{a}_{\mathbb{R}^T} f(x)\,dx = \Phi$$

Conclusion

- Transreal derivatives extend real derivatives
- Transreal integrals extend real integrals
- It is known that Newton's laws of motion extend to transarithmetic and transcalculus

Value

Reach and Reliability

- Transcalculus allows the solution of mathematical and physical problems at singularities
- Makes mathematical software more reliable

Transcalculus

is a *very* good idea