

Trans-Floating-Point Arithmetic

Dr James Anderson FBCS CITP CSci

Agenda

- Advantages of totality
- How to divide by zero
- How to double the range of real floats
- How to make relational operators independent
- Value to science and society

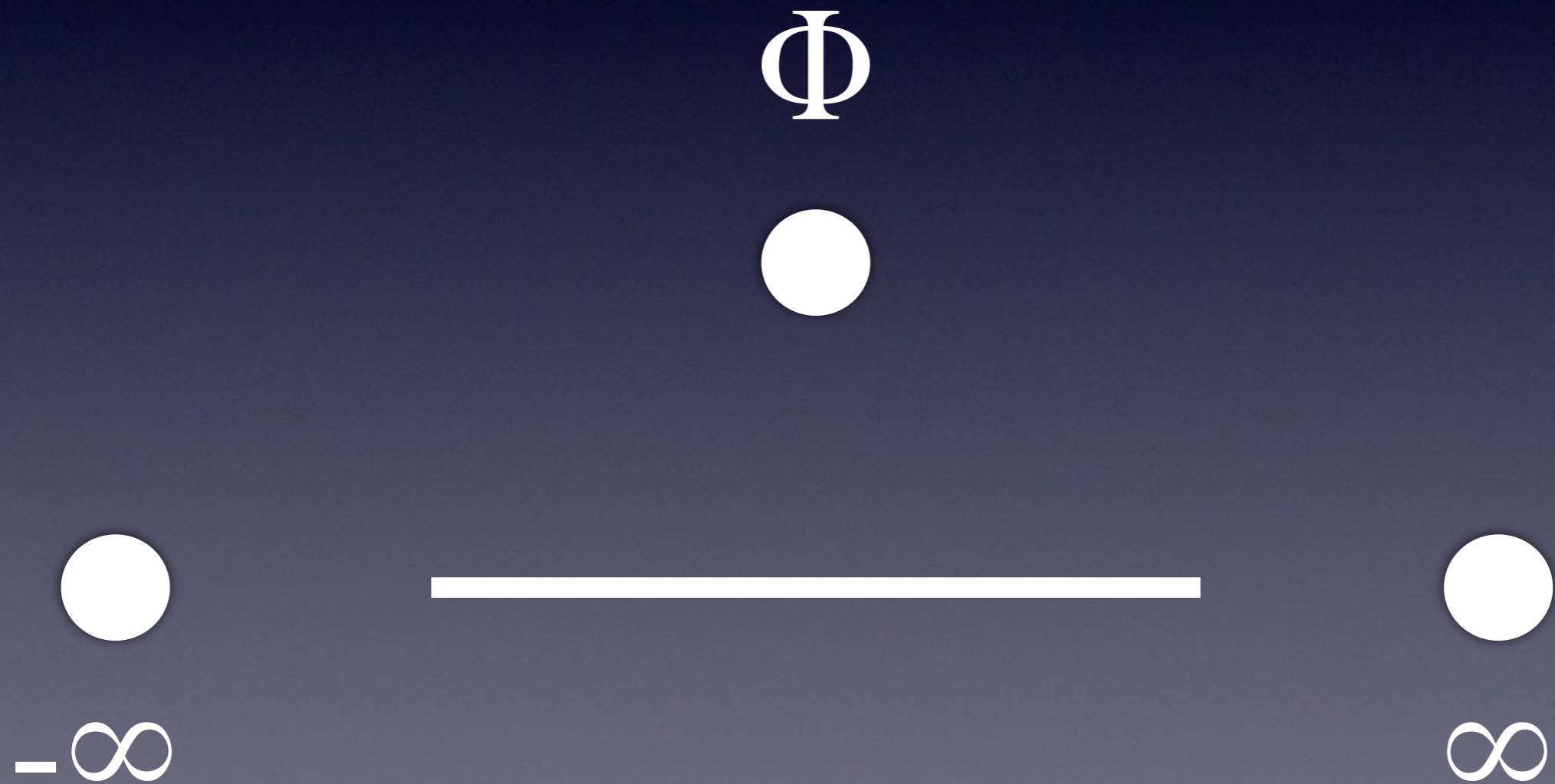
Advantages of Totality

Totallity

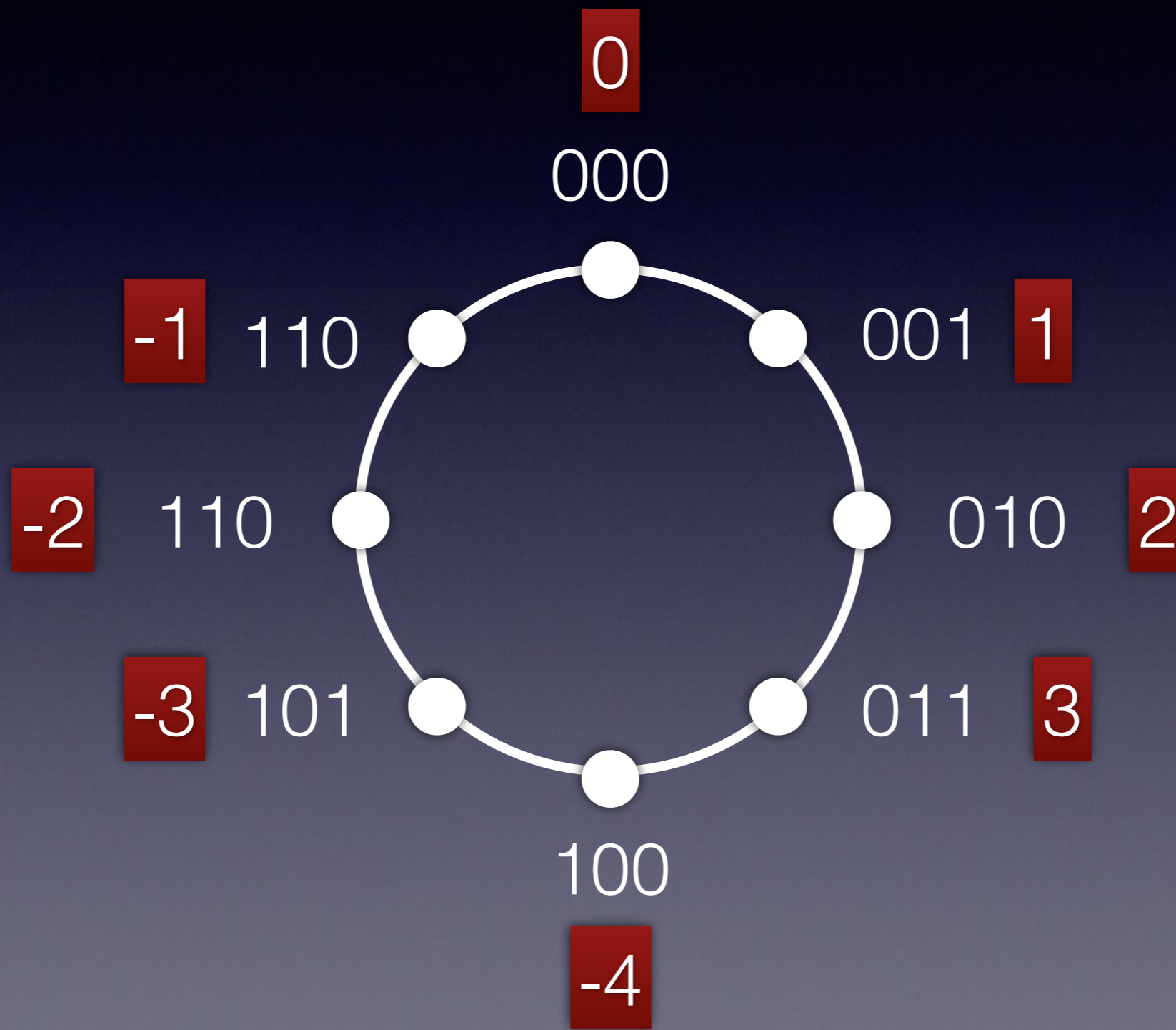
- Every operation can be applied to any arguments giving a valid result
- No logical exceptions - *ever!*
- Every syntactically correct sentence is semantically correct so no logical run-time errors - *ever!*

Dividing by Zero

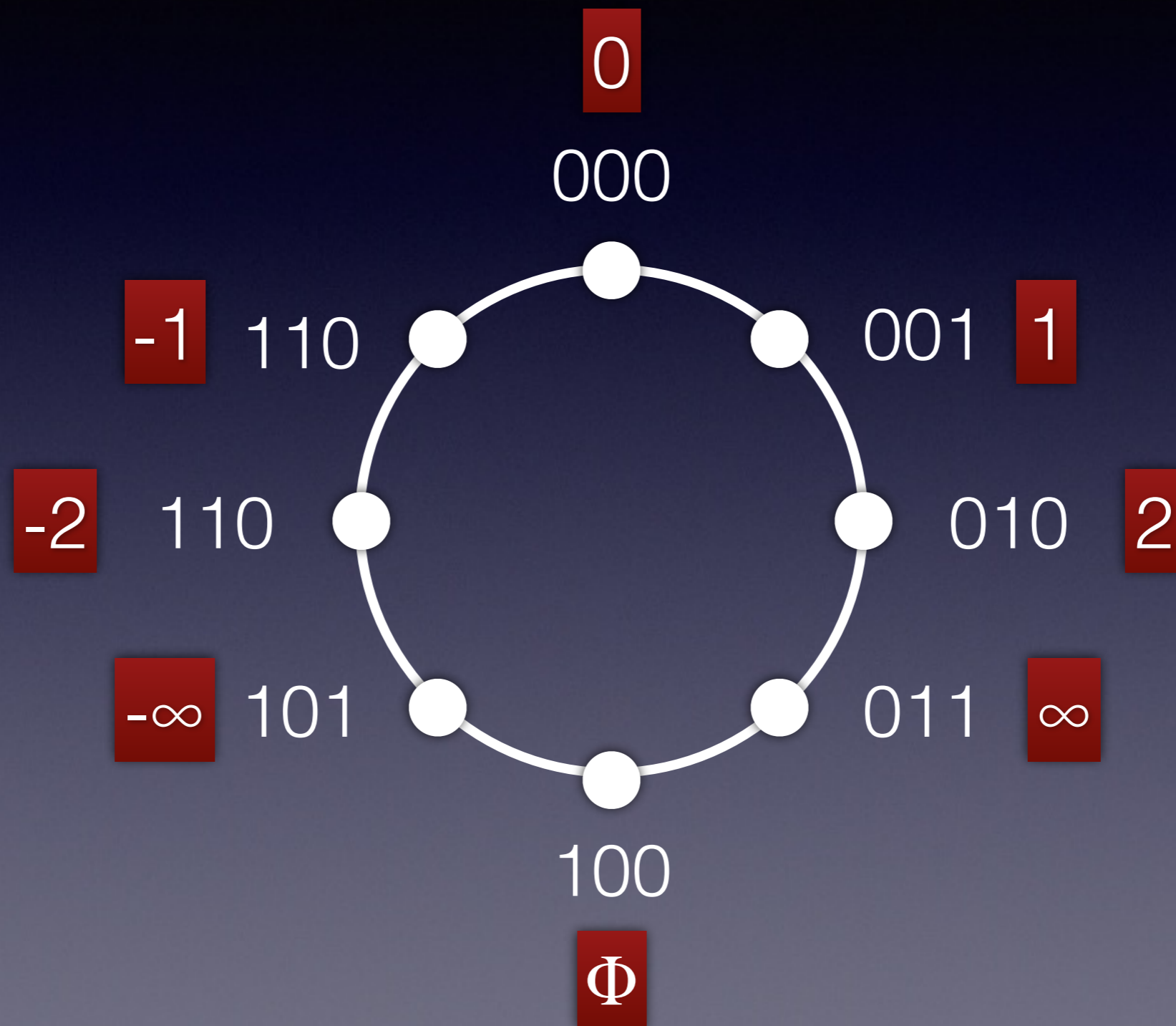
Transreal Number Line



Two's Complement



Trans Two's Complement



Transreal Numbers

Transreal numbers, t , are proper fractions of real numbers, with a non-negative denominator, d , and a numerator, n , that is one of $-1, 0, 1$ when $d = 0$

$$t = \frac{n}{d}$$

With k a positive constant:

$$-\infty = \frac{-k}{0} = \frac{-1}{0}$$

$$\Phi = \frac{0}{0}$$

$$\infty = \frac{k}{0} = \frac{1}{0}$$

Negative Denominators

An improper fraction may have a negative denominator ($-k$) which must be made positive *before* any arithmetical operator is applied

$$\frac{n}{-k} = \frac{-n}{-(-k)} = \frac{-1 \times n}{-1 \times (-k)} = \frac{-n}{k}$$

Multiplication

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Division

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

Addition of Two Infinities

$$\infty + \infty = \frac{1}{0} + \frac{1}{0} = \frac{1+1}{0} = \frac{2}{0} = \frac{1}{0} = \infty$$

$$\infty + (-\infty) = \frac{1}{0} + \frac{-1}{0} = \frac{1-1}{0} = \frac{0}{0} = \Phi$$

$$-\infty + \infty = \frac{-1}{0} + \frac{1}{0} = \frac{-1+1}{0} = \frac{0}{0} = \Phi$$

$$-\infty + (-\infty) = \frac{-1}{0} + \frac{-1}{0} = \frac{-1+(-1)}{0} = \frac{-2}{0} = \frac{-1}{0} = -\infty$$

General Addition

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Subtraction

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$$

Associativity

$$a + (b + c) = (a + b) + c$$

$$a \times (b \times c) = (a \times b) \times c$$

Commutativity

$$a + b = b + a$$

$$a \times b = b \times a$$

Partial Distributivity

$$a(b + c) = ab + ac$$

When $a \neq \pm\infty$ or

$bc > 0$ or

$$(b + c) / 0 = \Phi$$

Comparison

- Mathematics checks for division by zero and, if found, it fails
- Transmathematics checks for division by zero and always succeeds

Doubling Real Range

Floating-Point (NaN)

X 11111111111111 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

9,007,199,254,740,990

Trans-Float Zero and Nullity

X000000000000 00000000000000000000000000000000

000000000000000000000000000000000000000000000000000

Abolishing NaNs

- Doubles real range - giving larger numbers
- Or halves size of smallest, positive number - giving more accuracy

Independent Relational Operators

Floating-Point

- Four primitive, relational operators: $<$, $=$, $>$, $?$
- Said to be mutually independent
- So must be $2^4 = 16$ compound operators
- But there are only 11 compound operators so the standard is wrong!
- With negation, wastes 10 states

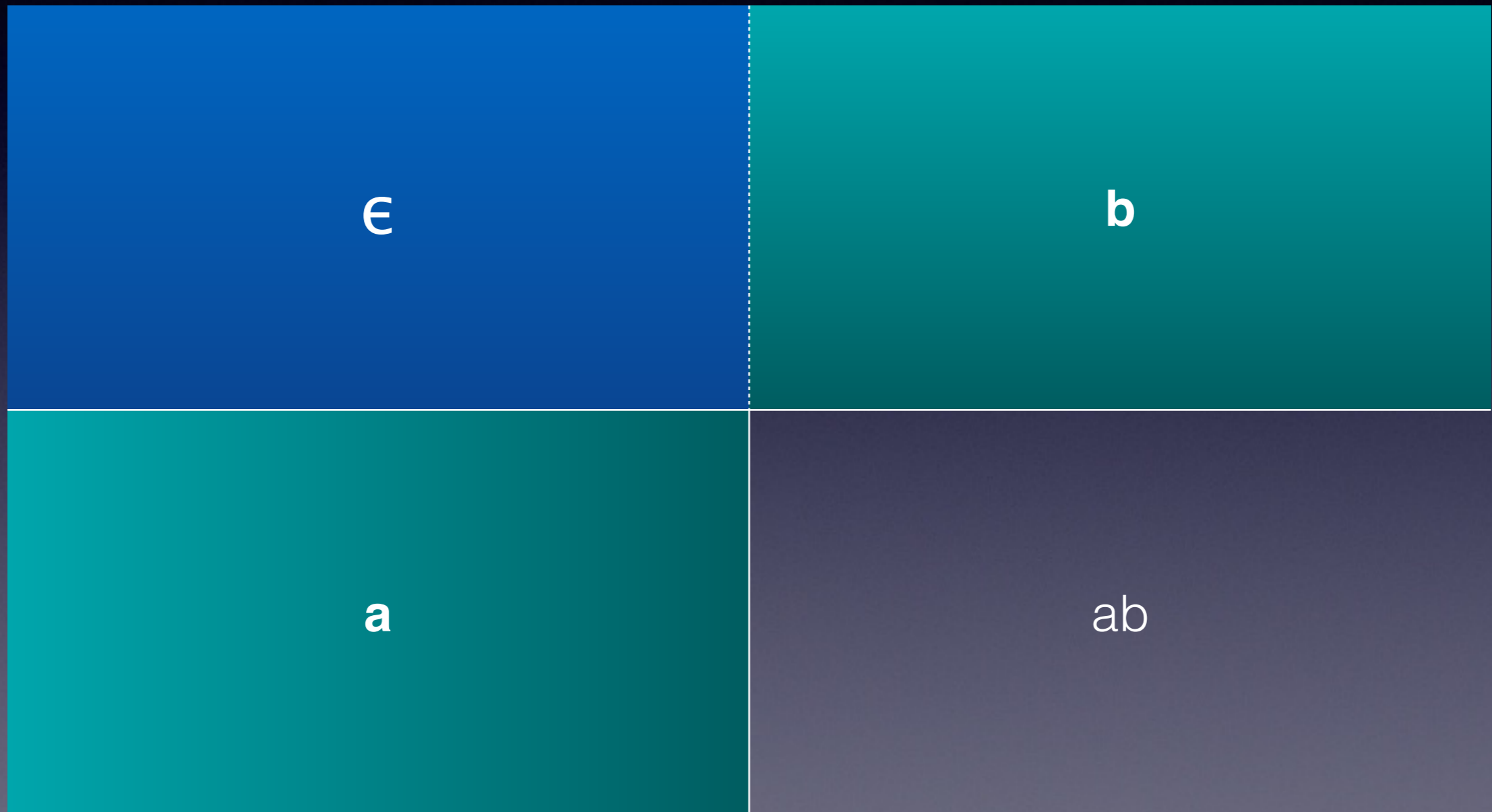
Mathematics

- Three primitive, relational operators: $<$, $=$, $>$
- Can be combined with not: $!$
- But all of the negations, except not-epsilon, are redundant so wastes 7 states

Trans Float/Mathematics

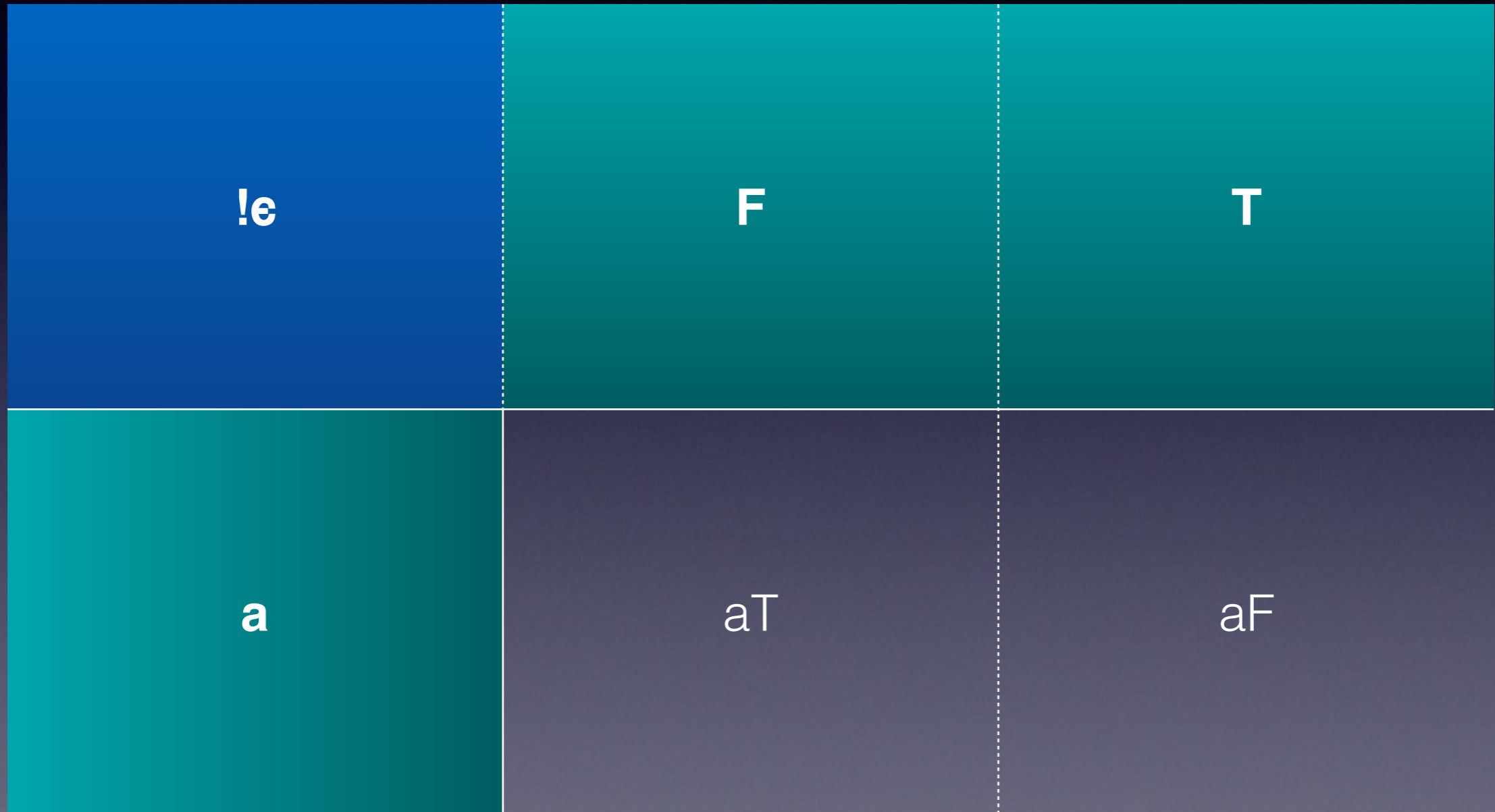
- Three primitive, relational operators: $<$, $=$, $>$
- Can be combined with not: $!$
- Giving $2^4 = 16$ compound operators
- Relational operators AND negation are independent

1 Epsilon



2

Not Epsilon



3 Less

<	$-\infty$	r2	∞	\emptyset
$-\infty$	F	T	T	F
r1	F	*	T	F
∞	F	F	F	F
\emptyset	F	F	F	F

4

Not Less

$!<$	$-\infty$	$r2$	∞	\emptyset
$-\infty$	T	F	F	T
$r1$	T	*	F	T
∞	T	T	T	T
\emptyset	T	T	T	T

5 Equal

=	$-\infty$	r2	∞	\emptyset
$-\infty$	T	F	F	F
r1	F	*	F	F
∞	F	F	T	F
\emptyset	F	F	F	T

6

Not Equal

\neq	$-\infty$	$r2$	∞	\emptyset
$-\infty$	F	T	T	T
$r1$	T	*	T	T
∞	T	T	F	T
\emptyset	T	T	T	F

7

Greater

$>$	$-\infty$	$r2$	∞	\emptyset
$-\infty$	F	F	F	F
$r1$	T	*	F	F
∞	T	T	F	F
\emptyset	F	F	F	F

8

Not Greater

$!>$	$-\infty$	$r2$	∞	Φ
$-\infty$	T	T	T	T
$r1$	F	*	T	T
∞	F	F	T	T
Φ	T	T	T	T

9

Less-or-Equal

\leq	$-\infty$	$r2$	∞	\emptyset
$-\infty$	T	T	T	F
$r1$	F	*	T	F
∞	F	F	T	F
\emptyset	F	F	F	T

10

Not Less-or-Equal

$!<=$	$-\infty$	$r2$	∞	\emptyset
$-\infty$	F	F	F	T
$r1$	T	*	F	T
∞	T	T	F	T
\emptyset	T	T	T	F

11

Less-or-Greater

$\langle \rangle$	$-\infty$	$r2$	∞	\emptyset
$-\infty$	F	T	T	F
$r1$	T	*	T	F
∞	T	T	F	F
\emptyset	F	F	F	F

12

Not Less-or-Greater

$!<>$	$-\infty$	$r2$	∞	ϕ
$-\infty$	T	F	F	T
$r1$	F	*	F	T
∞	F	F	T	T
ϕ	T	T	T	T

13

Equal-or-Greater

\Rightarrow	$-\infty$	$r2$	∞	\emptyset
$-\infty$	T	F	F	F
$r1$	T	*	F	F
∞	T	T	T	F
\emptyset	F	F	F	T

14

Not Equal-or-Greater

$\neq >$	$-\infty$	$r2$	∞	\emptyset
$-\infty$	F	T	T	T
$r1$	F	*	T	T
∞	F	F	F	T
\emptyset	T	T	T	F

15

Less-or-Equal-or-Greater

$\langle = \rangle$	$-\infty$	$r2$	∞	\emptyset
$-\infty$	T	T	T	F
$r1$	T	T	T	F
∞	T	T	T	F
\emptyset	F	F	F	T

16

Not Less-or-Equal-or-Greater

$!<=>$	$-\infty$	$r2$	∞	ϕ
$-\infty$	F	F	F	T
$r1$	F	F	F	T
∞	F	F	F	T
ϕ	T	T	T	F

Relops and Negation

- Floating-point wastes 10 states
- Mathematics wastes 7 states
- Trans Float/Mathematics waste 0 states

Total Order

- Trans Float/Mathematics is totally ordered in the extended-real numbers, with nullity as the uniquely unordered, transreal number
- IEEE 754 relational operator, TotalOrder, has the category error that a total ordering of bit patterns produces a total unorder of floating-point objects because $-\text{NaN}_i < F < \text{NaN}_i$ for all NaNs and all floating-point numbers F .

Value

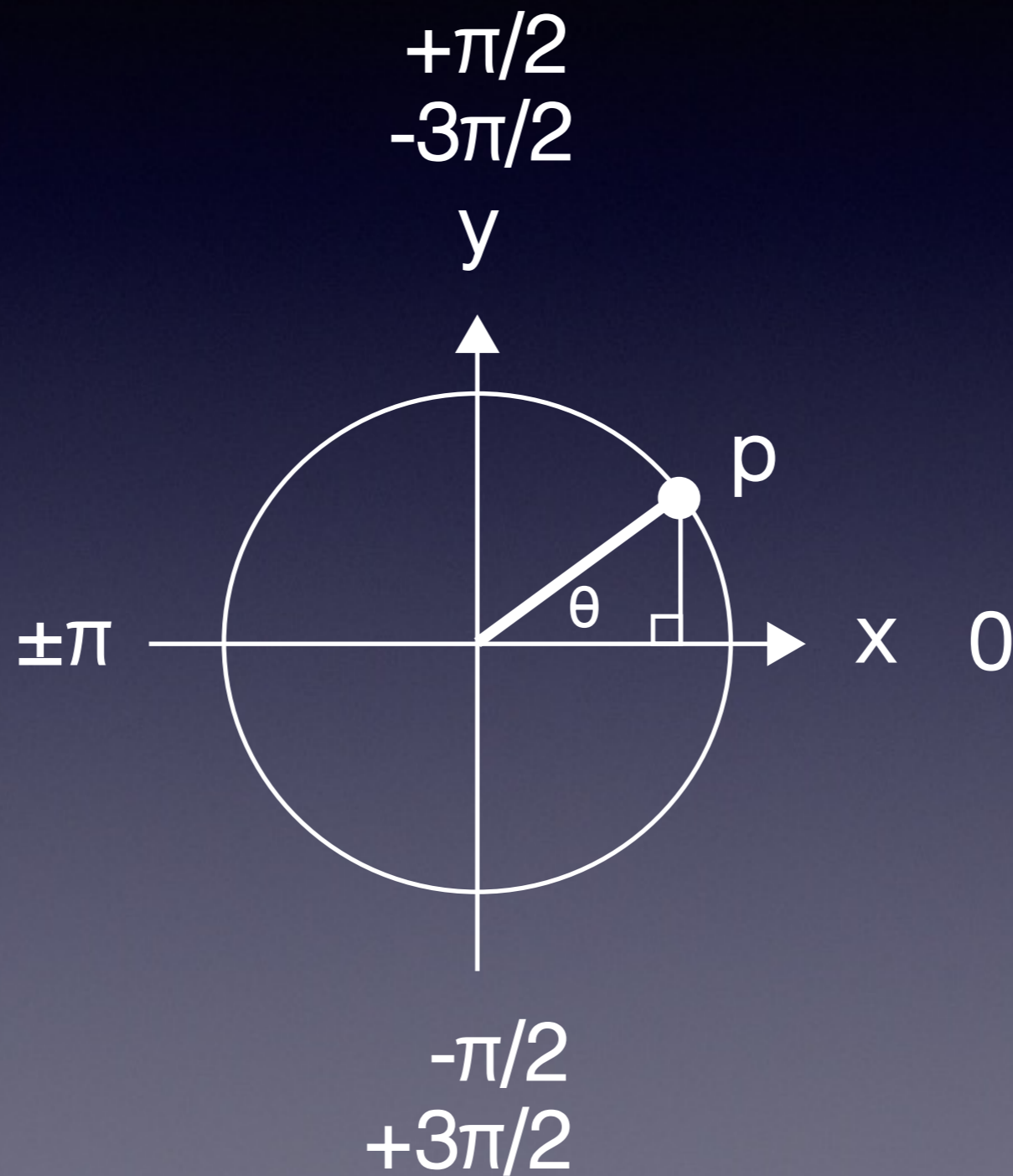
Trans-Floating-Point

- No wasted states
- No minus zero
- Double the real range or
Up to double the accuracy
- Independent relational and negation operators
- More reliable and accurate computation

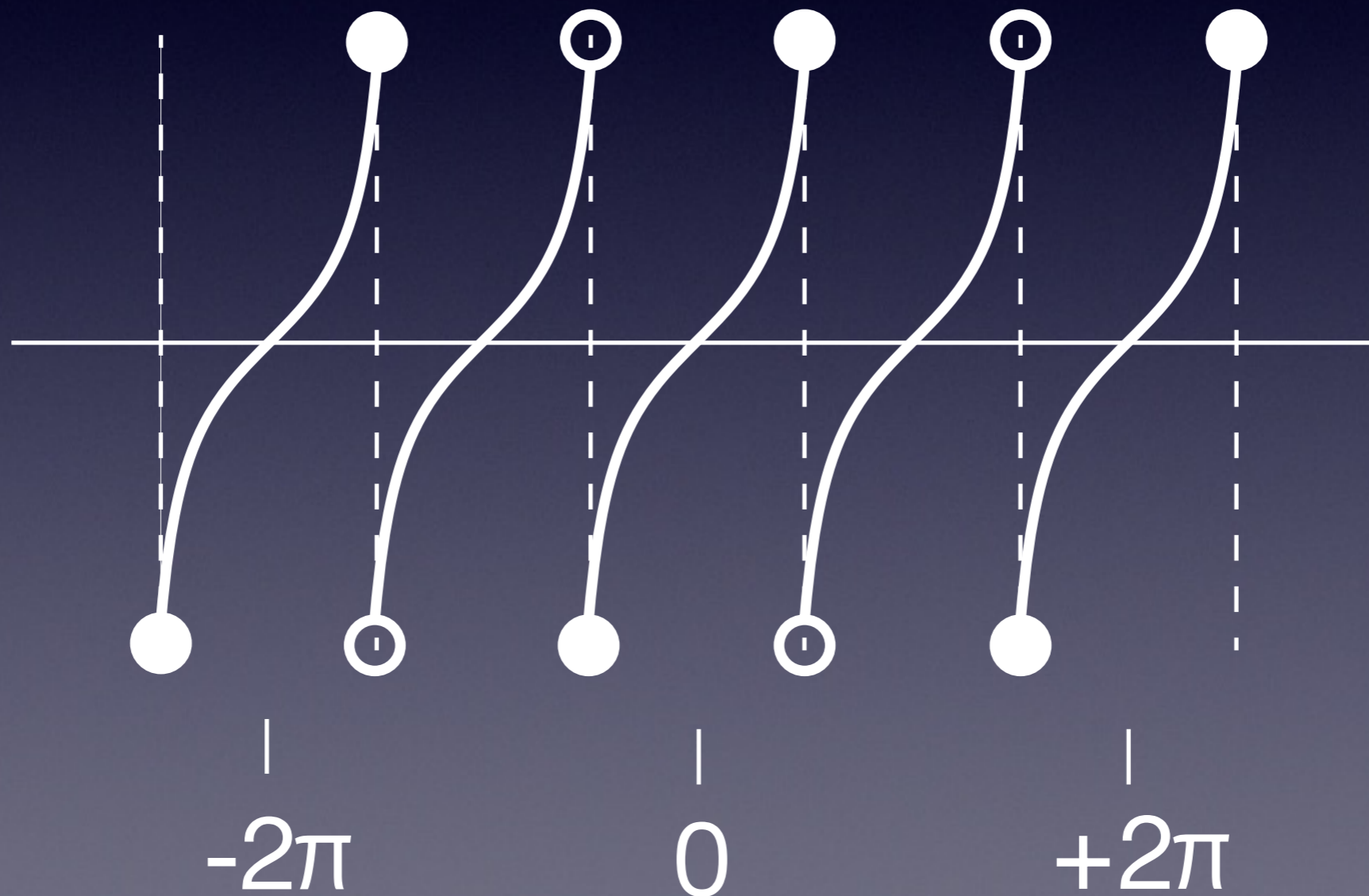
Compilation

- Every syntactically correct program is semantically correct
- No logical run-time errors - *ever!*
- Easier to verify safety-critical systems

Total Mathematics

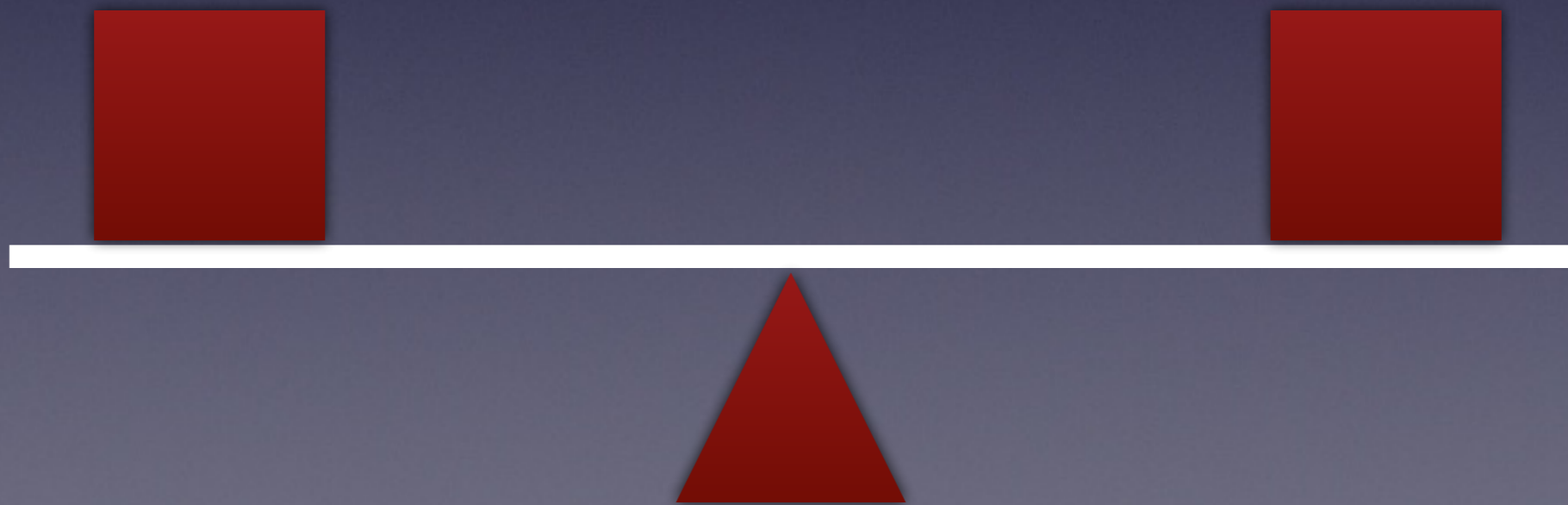


Total Mathematics



Total Physics

$$m_1 d_1 \leftarrow \rightleftarrows \rightarrow m_2 d_2$$



Consistency

- Two's Complement
- Floating-Point
- Transmathematics
- Transphysics
- More reliable and accurate applications

Trans-Floating-Point Arithmetic

is a *very* good idea!