Transreal Numbers

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Introduction

- Transreal numbers – what stopped you from dividing by zero for the last twelve hundred years?
- What is $\cos(\infty)$? What makes this question hard and the answer easy?
- Is $f(x) = 0/0$ continuous? What makes this question hard and the answer interesting?
- How do I use transreal numbers to make faster processors and safer software? What makes this question easy and the answer profitable?
- Conclusion - where do I go from here?

Algorithmic Basis

- Transreal arithmetic uses only the existing algorithms of arithmetic so it is consistent with real arithmetic, is easy to learn, and is easy to implement in both hardware and software

- Transreal arithmetic selects amongst the existing algorithms of arithmetic. The mechanism of selection delivers more mathematical content than real arithmetic so new things have to be learned

- Transreal arithmetic ignores the injunction not to divide by zero, thereby making it total – every arithmetical operation can be applied to any number and the result is a number. This abolishes all arithmetical exceptions

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Dividing by Zero

600 years ago no one could find the square root of any negative number, then someone defined the complex unit, $i = \sqrt{-1}$, and, after 400 years, everyone realised that mathematics continues to work well enough. Today complex numbers are widely accepted as both useful and mathematically interesting

Ten years ago I defined nullity, $\Phi = 0/0$, as a fixed, distinct number and adopted the definition of the signed infinities, $\pm\infty = \pm 1/0$, as fixed, distinct numbers. Since then I have given demonstrations that mathematics continues to work well enough. I am preparing to show that transreal numbers are useful

Dividing by Zero

- Dividing by zero is easy:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

- What stopped you from dividing by zero for the last twelve hundred years?

Cancellation

Cancel the largest, positive, common factor, if any, from the numerator and denominator. For example:

$$\frac{k}{0} = \frac{\operatorname{sgn}(k) \times |k|}{0 \times |k|} = \frac{\operatorname{sgn}(k)}{0} \in \left\{ \frac{-1}{0}, \frac{0}{0}, \frac{1}{0} \right\}$$

- Do primary schools in Norway teach cancellation correctly?
- Cancel these fractions: $2/2,\ 0/2,\ 2/0,\ 0/0$

Addition

Addition is harder than division:

- $\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (c \times b)}{b \times d}$ in general, but
- $\frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0}$ in particular

We could go on developing the algorithms of transreal arithmetic, but it is quicker to give its axioms

Axioms (Addition)

Additive Associativity: $a + (b + c) = (a + b) + c$ [A1]

Additive Commutativity: $a + b = b + a$ [A2]

Additive Identity: $0 + a = a$ [A3]

Additive Nullity: $\Phi + a = \Phi$ [A4]

Additive Infinity: $a + \infty = \infty : a \neq -\infty, \Phi$ [A5]

Axioms (Subtraction)

Subtraction as Sum with Opposite:

$a - b = a + (-b)$ [A6]

Bijectivity of Opposite: $-(-a) = a$ [A7]

Additive Inverse: $a - a = 0 : a \neq \pm\infty, \Phi$ [A8]

Opposite of Nullity: $-\Phi = \Phi$ [A9]

Non-null Subtraction of Infinity:

$a - \infty = -\infty : a \neq \infty, \Phi$ [A10]

Subtraction of Infinity from Infinity: $\infty - \infty = \Phi$ [A11]

Axioms (Multiplication)

Multiplicative Associativity:
$a \times (b \times c) = (a \times b) \times c$ [A12]

Multiplicative Commutativity: $a \times b = b \times a$ [A13]

Multiplicative Identity: $1 \times a = a$ [A14]

Multiplicative Nullity: $\Phi \times a = \Phi$ [A15]

Infinity Times Zero: $\infty \times 0 = \Phi$ [A16]

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Axioms (Division)

Division: $a \div b = a \times (b^{-1})$ [A17]

Multiplicative Inverse: $a \div a = 1 : a \neq 0, \pm\infty, \Phi$ [A18]

Bijectivity of Reciprocal: $(a^{-1})^{-1} = a : a \neq -\infty$ [A19]

Reciprocal of Zero: $0^{-1} = \infty$ [A20]

Reciprocal of the Opposite of Infinity:

$(-\infty)^{-1} = 0$ [A21]

Reciprocal of Nullity: $\Phi^{-1} = \Phi$ [A22]

Axioms (Ordering)

Positive: $\infty \times a = \infty \Leftrightarrow a > 0$ [A23]

Negative: $\infty \times a = -\infty \Leftrightarrow 0 > a$ [A24]

Positive Infinity: $\infty > 0$ [A25]

Ordering: $a - b > 0 \Leftrightarrow a > b$ [A26]

Less Than: $a > b \Leftrightarrow b < a$ [A27]

Greater Than or Equal: $a \geq b \Leftrightarrow (a > b) \vee (a = b)$ [A28]

Less Than or Equal: $a \leq b \Leftrightarrow b \geq a$ [A29]

Axioms (Quadrachotomy)

Quadrachotomy:

Exactly one of

$(a < 0)\ ,\ (a = 0)\ ,\ (a > 0)\ ,\ (a = \Phi)$ [A30]

Axioms (Distributivity)

Distributivity:

$$a \times (b + c) = (a \times b) + (a \times c) :$$

$$\neg((a = \pm\infty) \wedge (\operatorname{sgn}(b) \neq \operatorname{sgn}(c)) \wedge (b + c \neq 0, \Phi))$$

[A31]

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Axioms (Lattice Completeness)

Lattice Completeness:

The set, X, of all transreal numbers, excluding Φ, is lattice complete because

$$\forall Y : Y \subseteq X \Rightarrow (\exists u \in X : (\forall y \in Y : y \leq u) \wedge (\forall v \in X : (\forall y \in Y : y \leq v) \Rightarrow u \leq v)) \quad [A32]$$

Axioms

- The axioms of transreal arithmetic contain the axioms of real arithmetic
- The axioms of transreal arithmetic apply to all generalisations of real numbers
- The axioms of transreal arithmetic have been proved consistent by machine proof

Trigonometry

- What value does the function $\cos(\infty)$ take on?
- What makes this question hard?
- What makes this question easy?

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Trigonometry

$$\begin{aligned}
\cos(\infty) &= 1 - \frac{\infty^2}{2!} + \frac{\infty^4}{4!} - \ldots \\
&= \frac{1}{1} - \frac{1^2}{0^2 \times 2!} + \frac{1^4}{0^4 \times 4!} - \ldots \\
&= \frac{1}{1} - \frac{1}{0} + \frac{1}{0} - \ldots \\
&= \frac{1 \times 0 - 1 \times 1}{1 \times 0} + \frac{1}{0} - \ldots \\
&= \frac{-1}{0} + \frac{1}{0} - \ldots
\end{aligned}$$

Trigonometry

$$
\begin{aligned}
\cos(\infty) &= \frac{-1}{0} + \frac{1}{0} - \ldots \\
&= \frac{-1+1}{0} - \ldots \\
&= \frac{0}{0} - \ldots \\
&= \Phi - \ldots \\
&= \Phi
\end{aligned}
$$

Trigonometry

- $\exp(x) = \begin{cases} (\exp(-x))^{-1} & : x < 0 \\ \lim_{k \to \infty} 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} & : \text{otherwise} \end{cases}$
- $\exp(-\infty) = 0,\ \exp(\infty) = \infty,\ \exp(\Phi) = \Phi$
- $\cos^2 x + \sin^2 x = 1^x$
- $\cosh^2 x - \sinh^2 x = 1^x$

Continuity

Is $f(x) = 0/0$ continuous?

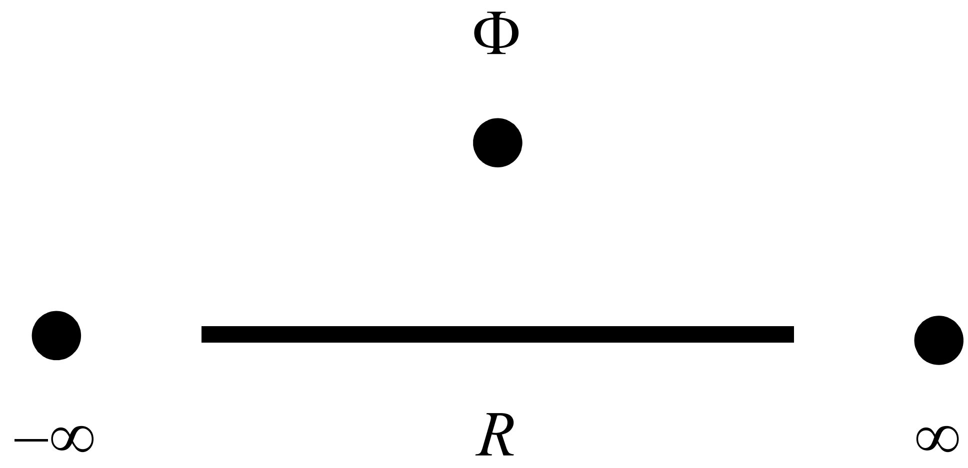
- What makes this question hard?

Topological Spaces

The open sets of the transreal numbers are generated from:

$$R, \{-\infty\}, \{\infty\}, \{\Phi\}$$

And can be visualised as:



Topological Spaces

Let $S_1 = \langle P_1, T_1 \rangle$ be a topological space over the transreal numbers with $P_1 = R^T = R \cup \{-\infty, \infty, \Phi\}$ and T_1 being the set of subsets of P_1

Let $S_2 = \langle P_2, T_2 \rangle$ be the topological space with $P_2 = \{\Phi\}$ and $T_2 = \{\Phi\} \cup \{\varnothing\}$

Now, $f : P_1 \to P_2$ is the total, constant function $f(x) = \Phi$ for all transreal x in P_1

Topological Spaces

First, if $U = \{\Phi\}$ then $U \in T_2$ and $f^{-1}(U) = R^T \in T_1$

Second, the trivial case, if $U = \{\varnothing\}$ then $U \in T_2$ and $f^{-1}(U) = \varnothing \in T_1$

This completes the proof that f is continuous

Similarly, the functions $f(x) = -\infty$ and $f(x) = \infty$ are continuous

Metric Spaces

Metric spaces are defined over a metric, m, which obeys four axioms:

$$m(a, b) = m(b, a) \qquad [M1]$$

$$m(a, b) \geq 0 \qquad [M2]$$

$$m(a, b) = 0 \Leftrightarrow a = b \qquad [M3]$$

$$m(a, b) + m(b, c) \geq m(a, c) \qquad [M4]$$

Replacing greater-than-or-equals with not-less-than generalises metric spaces to transmetric spaces

Transmetric Spaces

Transmetric spaces are defined over a transmetric, t, which obeys four axioms:

$$t(a, b) = t(b, a) \qquad [T1]$$

$$t(a, b) \nless 0 \qquad [T2]$$

$$t(a, b) = 0 \Leftrightarrow a = b \qquad [T3]$$

$$t(a, b) + t(b, c) \nless t(a, c) \qquad [T4]$$

Transmetric spaces contain metric spaces as a subset so limiting processes continue to work for the transreal numbers

Calculus

- $\lim_{x \to a} f(x) = l$ if for every real $\varepsilon > 0$ there is some real $\delta > 0$ such that, for all real x, if $0 < |x - a| < \delta$, then $|f(x) - l| < \varepsilon$

- $\lim_{x \to \infty} f(x) = l$ if for every real $\varepsilon > 0$ there is some real N such that, for all real $x > N$, it is the case that $|f(x) - l| < \varepsilon$

- $\lim_{x \to \infty} f(x) = \infty$ if for every real $\varepsilon > 0$ there is some real N such that, for all real $x > N$, it is the case that $f(x) > \varepsilon$

Calculus

No function can have a limit of Φ because:

- The distance from Φ is zero or else nullity, but zero has a fixed value and nullity is incommensurate with any other number so the distance can never be reduced in any process, let alone a limiting process
- Growing unboundedly is not moving in the direction of Φ
- By contrast a function can have a limit of ∞ or else $-\infty$ because growing unboundedly can move monotonically in the direction of ∞ or else $-\infty$

Regardless of any limit a function can have a value of Φ

Calculus

The function $f(x) = \Phi$ is continuous on $R^T \to \{\Phi\}$ despite the fact that $f(x) = \Phi$ does not have any limits

- What makes this interesting?

Software Engineering

- Software that performs all arithmetic in transreal numbers, or their generalisations, has no arithmetical exceptions
- Software that maps all language constructs, including memory management and peripheral handling, onto transreal numbers is total. That is, it has no exceptions
- Thus, transreal numbers make it easier to implement safety critical software

Processor Design

A transreal processor:

- Has no exceptional states
- Has no error handling circuitry
- Never stalls on error
- Is smaller and/or faster than a conventional processor
- Can be proved correct by counting through its states in a small design
- Can be proved correct by algebraic induction on a practically sized design

Where Next?

- Publish the work on topological spaces
- Fill out the mathematics between transreal arithmetic and topological spaces
- Construct better transreal processors
- Construct better transreal languages